Itô calculus and jump diffusions for G-Lévy processes

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Abstract

The paper considers the integration theory for G-Lévy processes with finite activity. We introduce the Itô-Lévy integrals, give the Itô formula for them and establish SDE's, BSDE's and decoupled FBSDE's driven by G-Lévy processes. In order to develop such a theory, we prove two key results: the representation of the sublinear expectation associated with a G-Lévy process and a characterization of random variables in $L^p_G(\Omega)$ in terms of their quasi-continuity.

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1. Introduction

In recent years much effort has been made to develop the theory of sublinear expectations connected with the volatility uncertainty and so-called G-Brownian motion. G-Brownian motion was introduced by Shige Peng in [8] as a way to incorporate the unknown volatility into financial models. Its theory is tightly associated with the uncertainty problems involving an undominated family of probability measures. Soon other connections have been discovered, not only in the field of financial mathematics, but also in the theory of path-dependent PDE's or 2BSDE's. Thus G-Brownian motion and connected G-expectation are very attractive mathematical objects.

Returning however to the original problem of volatility uncertainty in the financial models, one feels that G-Brownian motion is not sufficient to model the financial world, as both G- and the standard Brownian motion share the same property, which makes them often unsuitable for modelling, namely the continuity of paths. Therefore, it is not surprising that Hu and Peng introduced the process with jumps, which they called G-Lévy process (see [4]). Unfortunately, the theory of G-Lévy processes is still very undeveloped, especially compared G-Brownian motion. The follow-up has been limited to the paper by Ren ([11]), which introduces the representation of the sublinear expectation as an upper-expectation.

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In this paper we concentrate on establishing the integration theory for G-Lévy processes with finite activity. We introduce the integral w.r.t. the jump measure associated with the pure jump G-Lévy process (Section 5), give the Itô formula for general G-Itô-Lévy processes (Section 6) and we look at different typed of differential equations: both forward and backward (Section 7). The crucial piece of theory needed to obtain those results is the representation of the sublinear expectations, given in Section 3. The representation theorem, though inspired by the already quoted paper by Ren, also differs substantially, as we give the real representation of sublinear expectation connected with an arbitrary G-Lévy process with finite activity, whereas in the paper by Ren the opposite was shown: that the upper-expectation of the appropriate form induces some G-Lévy process. Another big difference in our approach is that we use different filtration (or rather the family of filtrations) for the sake of the proofs in the second part of this paper. Another result worth mentioning, is the complete characterization of the space $L^1_G(\Omega)$ in terms of (quasi)-continuity (Section 4).

2. Preliminaries

Let Ω be a given space and $\mathcal{H}$ be a vector lattice of real functions defined on Ω, i.e. a linear space containing 1 such that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. We will treat elements of $\mathcal{H}$ as random variables.

Definition 1. A sublinear expectation $\mathbb{E}$ *is a functional* $\mathbb{E}\colon \mathcal{H} \to \mathbb{R}$ *satisfying the following properties*

1. ***Monotonicity:*** *If* $X, Y \in \mathcal{H}$ *and* $X \geq Y$ *then* $\mathbb{E}[X] \geq \mathbb{E}[Y]$.
2. ***Constant preserving:*** *For all* $c \in \mathbb{R}$ *we have* $\mathbb{E}[c] = c$.
3. ***Sub-additivity:*** *For all* $X, Y \in \mathcal{H}$ *we have* $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y]$.
4. ***Positive homogeneity:*** *For all* $X \in \mathcal{H}$ *we have* $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall\, \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ *is called* a sublinear expectation space.

We will consider a space $\mathcal{H}$ of random variables having the following property: if $X_i \in \mathcal{H},\ i = 1, \dots n$ then

$$\phi(X_1, \dots, X_n) \in \mathcal{H}, \quad \forall\ \phi \in C_{b,Lip}(\mathbb{R}^n),$$

where $C_{b,Lip}(\mathbb{R}^n)$ is the space of all bounded Lipschitz continuous functions on $\mathbb{R}^n$. We will express the notions of a distribution and an independence of the random vectors using test functions in $C_{b,Lip}(\mathbb{R}^n)$.

Definition 2. *For an n-dimensional random vector $X = (X_1, \dots, X_n)$ define the functional $\mathbb{F}_X$ on $C_{b,Lip}(\mathbb{R}^n)$ as*

$$\mathbb{F}_X[\phi] := \mathbb{E}[\phi(X)], \quad \phi \in C_{b,Lip}(\mathbb{R}^n).$$

We will call the functional $\mathbb{F}_X$ the distribution of X. We say that two n-dimensional random vectors X_1 and X_2 (defined possibly on different sublinear expectation spaces) are identically distributed, if their distributions $\mathbb{F}_{X_1}$ and $\mathbb{F}_{X_2}$ are equal.

An m-dimensional random vector $Y = (Y_1, \dots, Y_m)$ is said to be independent of an n-dimensional random vector $X = (X_1, \dots, X_n)$ if

$$\mathbb{E}[\phi(X, Y)] = \mathbb{E}[\mathbb{E}[\phi(x, Y)]_{x=X}]. \quad \forall \phi \in C_{b,Lip}(\mathbb{R}^n \times \mathbb{R}^m).$$

Now we give the definition of G-Lévy process (after [4]).

Definition 3. *Let $X=(X_t)_{t\geq 0}$ be a d-dimensional càdlàg process on a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$. We say that X is a Lévy process if:*

1. *$X_0=0$,*
2. *for each $t,s\geq 0$ the increment $X_{t+s}-X_t$ is independent of $(X_{t_1},\ldots,X_{t_n})$ for every $n\in\mathbb{N}$ and every partition $0\leq t_1\leq\ldots\leq t_n\leq t$,*
3. *the distribution of the increment $X_{t+s}-X_t,\ t,s\geq 0$ is stationary, i.e. does not depend on t.*

Moreover, we say that a Lévy process X is a G-Lévy process, if satisfies additionally following conditions

4. *there a $2d$-dimensional Lévy process $(X^c_t,X^d_t)_{t\geq 0}$ such for each $t\geq 0$ $X_t=X^c_t+X^d_t$,*
5. *processes X^c and X^d satisfy the following growth conditions*

$$\lim_{t\downarrow 0}\mathbb{E}[|X^c_t|^3]t^{-1}=0;\quad \mathbb{E}[|X^d_t|]<Ct \text{ for all } t\geq 0.$$

Remark 1. The condition 5 implies that X^c is a d-dimensional generalized G-Brownian motion (in particular, it has continuous paths), whereas the jump part X^d is of finite variation.

Peng and Hu noticed in their paper that each G-Lévy process X might be characterized by a non-local operator G_X.

Theorem 2 (Lévy-Khintchine representation, Theorem 35 in [4])**.** *Let X be a d-dimensional G-Lévy process. For every $f\in C^3_b(\mathbb{R}^d)$ such that $f(0)=0$ we put*

$$G_X[f(.)]:=\lim_{\delta\downarrow 0}\mathbb{E}[f(X_\delta)]\delta^{-1}.$$

The above limit exists. Moreover, G_X has the following Lévy-Khintchine representation

$$G_X[f(.)]=\sup_{(v,p,Q)\in\mathcal{U}}\left\{\int_{\mathbb{R}^d\setminus\{0\}}f(z)v(dz)+\langle Df(0),q\rangle+\frac{1}{2}\operatorname{tr}[D^2f(0)QQ^T]\right\},$$

where $\mathcal{U}$ is a subset $\mathcal{U}\subset\mathcal{M}(\mathbb{R}^d\setminus\{0\})\times\mathbb{R}^d\times\mathbb{R}^{d\times d}$ and $\mathcal{M}(\mathbb{R}^d\setminus\{0\})$ is a set of all Borel measures on $(\mathbb{R}^d\setminus\{0\},\mathcal{B}(\mathbb{R}^d\setminus\{0\}))$. We know additionally that $\mathcal{U}$ has the property

$$\sup_{(v,p,Q)\in\mathcal{U}}\left\{\int_{\mathbb{R}^d\setminus\{0\}}|z|v(dz)+|q|+\operatorname{tr}[QQ^T]\right\}<\infty. \tag{1}$$

Theorem 3 (Theorem 36 in [4])**.** *Let X be a d-dimensional G-Lévy process. For each $\phi\in C_{b,Lip}(\mathbb{R}^d)$, define $u(t,x):=\mathbb{E}[\phi(x+X_t)]$. Then u is the unique viscosity solution of the following integro-PDE*

$$\begin{aligned}0=&\partial_t u(t,x)-G_X[u(t,x+.)-u(t,x)]\\ =&\partial_t u(t,x)-\sup_{(v,p,Q)\in\mathcal{U}}\left\{\int_{\mathbb{R}^d\setminus\{0\}}[u(t,x+z)-u(t,x)]v(dz)\right.\\ &\left.+\langle Du(t,x),q\rangle+\frac{1}{2}\operatorname{tr}[D^2u(t,x)QQ^T]\right\}\end{aligned}\tag{2}$$

with initial condition $u(0,x)=\phi(x)$.

It turns out that the set $\mathcal{U}$ used to represent the non-local operator G_X fully characterize X, namely having X we can define $\mathcal{U}$ satysfying eq. (1) and vice versa.

Theorem 4. *Let $\mathcal{U}$ satisfy (1). Consider the canonical probability space $\Omega := \mathbb{D}_0(\mathbb{R}^+, \mathbb{R}^d)$ of all càdlàg functions taking values in $\mathbb{R}^d$ equipped with the Skorohod topology. Then there exists a sublinear expectation $\hat{\mathbb{E}}$ on $\mathbb{D}_0(\mathbb{R}^+, \mathbb{R}^d)$ such that the canonical process $(X_t)_{t\geq 0}$ is a G-Lévy process satisfying Lévy-Khintchine representation with the same set $\mathcal{U}$.*

The proof might be found in [4] (Theorem 38 and 40). We will give however the construction of $\hat{\mathbb{E}}$, as it is important to understand it.

Begin with defining the sets of random variables. We denote $\Omega_T := \{\omega_{.\wedge T} \colon \omega \in \Omega\}$. Put

$$\begin{aligned} Lip(\Omega_T) :=&\{\xi \in L^0(\Omega) \colon \xi = \phi(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}), \\ &\phi \in C_{b,Lip}(\mathbb{R}^{d\times n}),\ 0 \leq t_1 < \ldots < t_n < T\}, \end{aligned}$$

where $X_t(\omega) = \omega_t$ is the canonical process on the space $\mathbb{D}_0(\mathbb{R}^+, \mathbb{R}^d)$ and $L^0(\Omega)$ is the space of all random variables, which are measurable to the filtration generated by the canonical process. We also set

$$Lip(\Omega) := \bigcup_{T=1}^{\infty} Lip(\Omega_T).$$

Firstly, consider the random variable $\xi = \phi(X_{t+s} - X_t)$, $\phi \in C_{b,Lip}(\mathbb{R}^d)$. We define

$$\hat{\mathbb{E}}[\xi] := u(s, 0),$$

where u is a unique viscosity solution of integro-PDE (2) with the initial condition $u(0, x) = \phi(x)$. For general

$$\xi = \phi(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}), \quad \phi \in C_{b,Lip}(\mathbb{R}^{d\times n})$$

we set $\hat{\mathbb{E}}[\xi] := \phi_n$, where ϕ_n is obtained via the following iterated procedure

$$\begin{aligned} \phi_1(x_1, \ldots, x_{n-1}) &= \hat{\mathbb{E}}[\phi(x_1, \ldots, X_{t_n} - X_{t_{n-1}})], \\ \phi_2(x_1, \ldots, x_{n-2}) &= \hat{\mathbb{E}}[\phi_1(x_1, \ldots, X_{t_{n-1}} - X_{t_{n-2}})], \\ &\vdots \\ \phi_{n-1}(x_1) &= \hat{\mathbb{E}}[\phi_{n-1}(x_1, X_{t_2} - X_{t_1})], \\ \phi_n &= \hat{\mathbb{E}}[\phi_{n-1}(X_{t_1})]. \end{aligned}$$

Lastly, we extend definition of $\hat{\mathbb{E}}[.]$ on the completion of $Lip(\Omega_T)$ (respectively $Lip(\Omega)$) under the norm $\|.\|_p := \hat{\mathbb{E}}[|.|^p]^{1/p}$, $p \geq 1$. We denote such a completion by $L^p_G(\Omega_T)$ (or resp. $L^p_G(\Omega)$).

It is also important to note that using this procedure we may in fact define the conditional sublinear expectation $\hat{\mathbb{E}}[\xi|\Omega_t]$. Namely, w.l.o.g. we may assume that $t = t_i$ for some i and then

$$\hat{\mathbb{E}}[\xi|\Omega_{t_i}] := \phi_{n-i}(X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_i} - X_{t_{i-1}}).$$

One can easily prove that such an operator is continuous w.r.t. the norm $\|.\|_1$ and might be extended to the whole space $L^1_G(\Omega)$. By construction above, it is clear that the conditional expectation satisfies the tower property, i.e. is dynamically consistent.

3. Representation of $\hat{\mathbb{E}}[.]$ as an upper-expectation

In the rest of this paper we will work on the canonical probability space $\Omega := \mathbb{D}_0(\mathbb{R}^+, \mathbb{R}^d)$ and the sublinear expectation $\hat{\mathbb{E}}[.]$ such that the canonical process X is a G-Lévy process satisfying the Lévy-Khintchine representation for some set $\mathcal{U}$. Just as in the case of G-Brownian motion it is reasonable to ask, if we can represent $\hat{\mathbb{E}}[.]$ as an upper-expectation (i.e. supremum of expectations related to the probability measures on $\mathbb{D}_0(\mathbb{R}^+, \mathbb{R}^d)$). Moreover, can we describe these probability measures as laws of some processes on $\mathbb{D}_0(\mathbb{R}^+, \mathbb{R}^d)$?

These questions have been partially addressed in the paper by Liying Ren [11]. He showed that for every $\mathcal{U}$ there exist a relatively compact family of probability measures $\mathfrak{P}$ such that

$$\hat{\mathbb{E}}[\xi] = \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}^{\mathbb{P}}[\xi], \quad \xi \in Lip(\Omega).$$

He also proved that if we take the laws of the Itô-Lévy integrals w.r.t. some Lévy process and an appropriate set of integrands, then upper-expectation w.r.t. this family of law makes a canonical process the G-Lévy process with some characteristics $\mathcal{U}$. This result is very interesting, but one would rather have an opposite result: having the sublinear expectation $\hat{\mathbb{E}}[.]$ one would like to define the set of integrands such that the laws of the Itô -Lévy integrals of this integrands give the representation of $\hat{\mathbb{E}}[.]$ as an upper-expectation.

The aim of this section is to introduce such a representation. As in the original paper [3], which represents a G-expectation as an upper-expectation, we will use the dynamic programming method and we will prove first the dynamic programming principle (DPP) in our set-up. What is important is that we will prove DPP with a different filtration than it is classically used. This change of filtration is both important for proving DPP (though it is seems that it might be also proved with the canonical filtration) and for establishing Itô calculus. We will return to this issue at the appropriate moments.

Assumption 1. *Let a canonical process X be a G-Lévy process in $\mathbb{R}^d$ on a sublinear expectation space $(\mathbb{D}_0(\mathbb{R}^+, \mathbb{R}^d), L^1_G(\Omega), \hat{\mathbb{E}})$. Assume that the set $\mathcal{U}$ used in the Lévy-Khintchine representation has the product form $\mathcal{U} = \mathcal{V}\times\mathcal{P}\times\mathcal{Q} \subset \mathcal{M}(\mathbb{R}^d\setminus\{0\})\times\mathbb{R}^d\times\mathbb{R}^{d\times d}$. Moreover, assume that $\mathcal{V}$ is a set of Lévy measures satisfying following condition*

$$\lambda := \sup_{v\in\mathcal{V}} v(\mathbb{R}^d \setminus \{0\}) < \infty.$$

*Such a process will be called a G-*Lévy process with finite activity*. We will also assume without loss of generality that $\lambda = 1$ (otherwise we need to adjust the intensity of Poisson procress, which will be introduced in a moment) and that every measure in $\mathcal{V}$ is a probability measure on $\mathbb{R}^d$ (otherwise we add an atom in 0).*

On the same canonical space introduce now a probability measure $\mathbb{P}$ and processes W and N such that W is a $\mathbb{P}$-standard Brownian motion in $\mathbb{R}^d$ and N is a $\mathbb{P}$-standard

1-dimensional Poisson process with intensity 1, which is independent of W. Let $\mathcal{N}$ be collection of $\mathbb{P}$-null sets and

$$\begin{aligned}
\mathcal{F}_t :=& \sigma\{W_s,\ N_s\colon 0\le s\le t\}\vee\mathcal{N};\\
\mathbb{F} :=& (\mathcal{F}_t)_{t\ge 0};\\
\mathcal{F}_t^s :=& \sigma\{W_u - W_s,\ N_u - N_s\colon s\le u\le t\}\vee\mathcal{N}\quad 0\le s\le t;\\
\mathbb{F}^s :=& (\mathcal{F}_t^s)_{t\ge s}\quad s\ge 0.
\end{aligned}$$

It is important to note that in fact we will work with the family of filtrations $(\mathbb{F}^s)_{s\ge 0}$ indexed with time s. This is reflected in the following definition

Definition 4. *Introduce the set of integrands $\mathcal{A}^{\mathcal{U}}_{t,T}$ associated with $\mathcal{U}$ as follows*

$$\begin{aligned}
\mathcal{A}^{\mathcal{U}}_{t,T} := \{\theta = (\theta^{1,c},\theta^{2,c},\theta^d)\colon\ & (\theta^{1,c},\theta^{2,c}) \text{ is an } \mathbb{F}^t\text{-adapted process on }]t,T]\\
& \text{taking values in } \mathcal{P}\times\mathcal{Q} \text{ and } \theta^d \text{ is an } \mathbb{F}^t\text{-predictable process on }]t,T]\\
& \text{such that the distribution } \mu_{\theta^d_s}\in\mathcal{V},\ \forall s>t\}.
\end{aligned}$$

We stress that we use filtration $\mathbb{F}^t$ instead of $\mathbb{F}$! We also note that if $]t_1,T_1]\subset]t_2,T_2]$ then $\mathcal{A}^{\mathcal{U}}_{t_1,T_1}\subset\mathcal{A}^{\mathcal{U}}_{t_2,T_2}$.

For $\theta\in\mathcal{A}^{\mathcal{U}}_{0,\infty}$ denote the following Lévy -Itô integral as

$$B_T^{t,\theta} = \int_t^T \theta_s^{1,c}ds + \int_t^T \theta_s^{2,c}dW_s + \int_t^T \theta_s^d dN_s.$$

Lastly, for a fixed $\phi\in C_{b,Lip}(\mathbb{R}^d)$ and fixed $T>0$ define for each $(t,x)\in[0,T]\times\mathbb{R}^d$

$$u(t,x) := \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,T}} \mathbb{E}^{\mathbb{P}}[\phi(x+B_T^{t,\theta})].$$

The most crucial results of this section are to be found in these three theorems.

Theorem 5 (DPP)**.** *For every $h>0$ such that $t+h<T$ one has*

$$u(t,x) = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,t+h}} \mathbb{E}^{\mathbb{P}}[u(t+h,x+B_{t+h}^{t,\theta})].$$

Theorem 6. *u is the viscosity solution of the following integro-PDE*

$$\partial_t u(t,x) + G_X[u(t,x+.)-u(t,x)] = 0$$

with the terminal condition $u(T,x)=\phi(x)$.

Theorem 7. *Let $\xi\in Lip(\Omega_T)$ has the representation*

$$\xi = \phi(X_{t_1}, X_{t_2}-X_{t_1},\ldots,X_{t_n}-X_{t_{n-1}}),\quad \phi\in C_{b,Lip}(\mathbb{R}^{d\times n}).$$

Then

$$\begin{aligned}
\hat{\mathbb{E}}[\xi] &= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,\infty}} \mathbb{E}^{\mathbb{P}}[\phi(B_{t_1}^{0,\theta}, B_{t_2}^{t_1,\theta},\ldots,B_{t_n}^{t_{n-1},\theta})]\\
&= \sup_{\theta^1\in\mathcal{A}^{\mathcal{U}}_{0,t_1}}\ \sup_{\theta^2\in\mathcal{A}^{\mathcal{U}}_{t_1,t_2}}\ \ldots\ \sup_{\theta^n\in\mathcal{A}^{\mathcal{U}}_{t_{n-1},t_n}} \mathbb{E}^{\mathbb{P}}[\phi(B_{t_1}^{0,\theta^1}, B_{t_2}^{t_1,\theta^2},\ldots,B_{t_n}^{t_{n-1},\theta^n})].
\end{aligned}$$

Note that due to working with the family of filtrations $\mathbb{F}^t$, $t \geq 0$, the last equality is non-trivial (and extremely useful). For the sake of the shortness of notation, we will also get rid of the chain of supremums by defining

$$\mathcal{A}_\pi^\mathcal{U} := \{\theta = (\theta_s)_{s\in]t_1,t_n]} \colon \theta|_{]t_k,t_{k+1}]} \in \mathcal{A}^\mathcal{U}_{t_k,t_{k+1}},\ k = 1,\dots,n-1\},$$

where $\pi := \{t_1,\dots,t_n\}$ is a partition of $]0,T]$. Then the last equality in Theroem 7 might be simply written as

$$\hat{\mathbb{E}}[\xi] = \sup_{\theta\in\mathcal{A}_\pi^\mathcal{U}} \mathbb{E}^\mathbb{P}[\phi(B^{0,\theta}_{t_1}, B^{t_1,\theta}_{t_2},\dots,B^{t_{n-1},\theta}_{t_n})].$$

Lastly, we will give the following easy corollary to Theorem 7.

Corollary 8. *Let $\xi \in L^1_G(\Omega)$. Then for any partition π of the interval $]0,\infty[$ one has*

$$\hat{\mathbb{E}}[\xi] = \sup_{\theta\in\mathcal{A}_\pi^\mathcal{U}} \mathbb{E}^\mathbb{P}[\xi \circ B_\cdot^{0,\theta}],$$

where $\circ$ denotes composition of functions. We treat here the Itô integral $B_\cdot^{0,\theta}$ as a function of Ω (i.e. the space of càdlàg functions) taking values in càdlàg functions. We can also write it in a different form, defining $\mathbb{P}^\theta := \mathbb{P}\circ(B_\cdot^{0,\theta})^{-1}$. Then

$$\hat{\mathbb{E}}[\xi] = \sup_{\theta\in\mathcal{A}_\pi^\mathcal{U}} \mathbb{E}^{\mathbb{P}^\theta}[\xi].$$

We will give here only the proof of Theorem 5, leaving the proofs of Theorems 6 and 7 to the Appendix, as they are very similar to proofs already published, which can be found in both [3] and [11]. The proof of Theorem 5 also follows the idea by Denis *et al.* however one has to change details due to the changed filtration. One also needs to take into consideration the fact that Lemma 41 in their paper is usually untrue because of strict conditions on the distribution of θ^d. However, the proof retains similar structure, i.e. we firstly prove some lemmas which gives us control over appropriate essential supremums.

Just as Denis *et al.* for $\zeta \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^n)$, $t\in[0,T]$ and a fixed $\phi \in C_{b,Lip}(\mathbb{R}^n\times\mathbb{R}^d)$ put

$$\Lambda_{t,T}[\zeta] := \operatorname*{ess\,sup}_{\theta\in\mathcal{A}^\mathcal{U}_{t,T}} \mathbb{E}^\mathbb{P}[\phi(\zeta, B^{t,\theta}_T)|\mathcal{F}_t].$$

Lemma 9. *$\Lambda_{t,T}[.] \colon L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^n) \to L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R})$ is bounded by the bound of ϕ and Lipschitz continuous with the same constant as ϕ, i.e.*

1. *$\Lambda_{t,T}[\zeta] \leq C_\phi$,*
2. *$|\Lambda_{t,T}[\zeta] - \Lambda_{t,T}[\zeta']| \leq L_\phi|\zeta-\zeta'|$.*
3. *$\Lambda_{t,T}[x]$ is deterministic for every $x\in\mathbb{R}^n$ and $\Lambda_{t,T}[x] = \Lambda_{0,T-t}[x]$.*

Proof. Points 1 and 2 follow exactly the same argument as in Lemma 42 of [3]. The first assertion of point 3 follows from the fact that $B^{t,\theta}_T$ is independent of $\mathcal{F}_t$ for every $\theta\in\mathcal{A}^\mathcal{U}_{t,T}$ due to the choice of filtration, thus

$$\mathbb{E}^\mathbb{P}[\phi(x,B^{t,\theta}_T)|\mathcal{F}_t] = \mathbb{E}^\mathbb{P}[\phi(x,B^{t,\theta}_T)].$$

The last assertion is an easy consequence of the fact that the law of $B^{t,\theta}_T,\ \theta\in\mathcal{A}^\mathcal{U}_{t,T}$ is the same as the law of $B^{0,\tilde{\theta}}_{T-t}$, where $\tilde{\theta}\in\mathcal{A}^\mathcal{U}_{0,T-t}$ is defined by $\tilde{\theta}_s := \theta_{t+s},\ s\in]0,T-t]$. $\square$

To analyse better the essential supremums, we note that following general lemma holds.

Lemma 10. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{A_k\}_{k=1}^N$ be an $\mathcal{F}$-partition of Ω. Take the families of random variables $\{X_i, k\}_{i\in I} \subset L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, $k \in 1, \ldots, N$. Then*

$$\sum_{k=1}^N \mathbb{1}_{A_k} \operatorname*{ess\,sup}_{i\in I} X_{i,k} = \operatorname*{ess\,sup}_{i\in I} \sum_{k=1}^N \mathbb{1}_{A_k} X_{i,k}, \quad \mathbb{P} - a.s.$$

The proof of the lemma will be given in Appendix.

Lemma 11. *Let $u_{t,T}(x) := \Lambda_{t,T}[x]$, $x \in \mathbb{R}^n$. Then for every $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$ one has*

$$u_{t,T}(\zeta) = \Lambda_{t,T}[\zeta] \quad \mathbb{P} - a.s.$$

Proof. By the Lipschitz continuity of $\Lambda_{t,T}$ it is sufficient to prove the assertion for $\zeta = \sum_{k=1}^N x_k \mathbb{1}_{A_k}$, where $\{A_k\}_{k=1}^N$ is an $\mathcal{F}_t$-partition of Ω. By definition of $\Lambda_{t,T}$ and Lemma 10 one has $\mathbb{P}$-a.s.

$$\begin{aligned}
\Lambda_{t,T}[\zeta] &= \operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,T}} \mathbb{E}^{\mathbb{P}}[\phi(\zeta, B_T^{t,\theta})|\mathcal{F}_t] = \operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,T}} \mathbb{E}^{\mathbb{P}}[\phi(\sum_{k=1}^N x_k \mathbb{1}_{A_k}, B_T^{t,\theta})|\mathcal{F}_t] \\
&= \operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,T}} \sum_{k=1}^N \mathbb{1}_{A_k} \mathbb{E}^{\mathbb{P}}[\phi(x_k, B_T^{t,\theta})|\mathcal{F}_t] = \sum_{k=1}^N \mathbb{1}_{A_k} \operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,T}} \mathbb{E}^{\mathbb{P}}[\phi(x_k, B_T^{t,\theta})|\mathcal{F}_t] \\
&= \sum_{k=1}^N \mathbb{1}_{A_k} u_{t,T}(x_k) = u_{t,T}(\zeta).
\end{aligned}$$

□

Unfortunately, working with the family of filtrations, instead of only one filtration, makes us introduce another operator, which would deal with the integrands, which are adapted or predictable (respectively) to an 'earlier' filtration. Namely, let $s < t < T$. Define a set

$$\mathcal{A}^{\mathcal{U}}_{s,t,T} := \{\theta = (\theta_u)_{u\in[t,T]} : \theta = \tilde{\theta}|_{]t,T]} \text{ for some } \tilde{\theta} \in \mathcal{A}^{\mathcal{U}}_{s,T}\}$$

and a function

$$\Lambda_{s,t,T}[\zeta] := \operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{s,t,T}} \mathbb{E}^{\mathbb{P}}[\phi(\zeta, B_T^{t,\theta})|\mathcal{F}_t], \quad \zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n) \text{ and } \phi \in C_{b,Lip}(\mathbb{R}^n \times \mathbb{R}^d).$$

Similarly, we have the following proposition

Lemma 12. 1. *$\Lambda_{s,t,T}$ is bounded and Lipschitz continuous.*

2. *$\Lambda_{s,t,T}[x]$ is deterministic and $\Lambda_{s,t,T}[x] = \Lambda_{t,T}[x] = u_{t,T}(x)$ for each $x \in \mathbb{R}^n$.*
3. *$\Lambda_{s,t,T}[\zeta] = u_{t,T}(\zeta)(= \Lambda_{t,T}[\zeta])$ for each $\zeta \in L^2(\Omega, \mathcal{F}^s_t, \mathbb{P}; \mathbb{R}^n)$.*

Proof. Point 1 follows the same argument as in Lemma 9. Let us prove property 2. Just as in the proof of Lemma 43 in [3], we argue that the collection of processes

$$\mathcal{A} := \{\theta = \sum_{j=1}^{N} \mathbb{1}_{A_j}\theta^j : \ \{A_j\}_{j=1}^N \text{ is an } \mathcal{F}_t^s\text{-partition of } \Omega, \ \theta^j \in \mathcal{A}_{t,T}^{\mathcal{U}}\}$$

is dense in $\mathcal{A}_{s,t,T}^{\mathcal{U}}$, so by the Lipschitz continuity of both stochastic integral and ϕ we can take the essential supremum over this dense set. Thus

$$\begin{aligned}
\Lambda_{s,t,T}[x] &= \operatorname*{ess\,sup}_{\theta\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}[\phi(x, B_T^{t,\theta})|\mathcal{F}_t] \\
&= \operatorname*{ess\,sup}_{\substack{\{A_j\}_{j=1}^N:\ \mathcal{F}_t^s\text{-partition of } \Omega \\ \theta^j\in\mathcal{A}_{t,T}^{\mathcal{U}}}} \mathbb{E}^{\mathbb{P}}[\phi(x, B_T^{t,\sum_{j=1}^N \mathbb{1}_{A_j}\theta^j})|\mathcal{F}_t] \\
&= \operatorname*{ess\,sup}_{\substack{\{A_j\}_{j=1}^N:\ \mathcal{F}_t^s\text{-partition of } \Omega \\ \theta^j\in\mathcal{A}_{t,T}^{\mathcal{U}}}} \sum_{j=1}^{N} \mathbb{1}_{A_j}\mathbb{E}^{\mathbb{P}}[\phi(x, B_T^{t,\theta^j})|\mathcal{F}_t] \\
&= \operatorname*{ess\,sup}_{\substack{\{A_j\}_{j=1}^N:\ \mathcal{F}_t^s\text{-partition of } \Omega \\ \theta^j\in\mathcal{A}_{t,T}^{\mathcal{U}}}} \sum_{j=1}^{N} \mathbb{1}_{A_j}\mathbb{E}^{\mathbb{P}}[\phi(x, B_T^{t,\theta^j})].
\end{aligned}$$

Taking the partition $A_1 = \Omega$ one can see that $\Lambda_{s,t,T}[x] \geq \Lambda_{t,T}[x]$. But of course each $\mathbb{E}^{\mathbb{P}}[\phi(x, B_T^{t,\theta^j})] \leq \Lambda_{t,T}[x]$ so also essential supremum is less or equal to $\Lambda_{t,T}[x]$. We conclude that $\Lambda_{s,t,T}[x] = \Lambda_{t,T}[x]$. Property 3 follows from Lemma 10, just as in Lemma 11. □

The following proposition proves that there is 'dynamic consistency' in taking the essential supremum.

Proposition 13. *Let $\xi \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^n)$, $\psi \in C_{b,Lip}(\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^d)$ and $0 \leq s < t < T$. Then*

$$\operatorname*{ess\,sup}_{\theta\in\mathcal{A}_{s,T}^{\mathcal{U}}} \mathbb{E}^{\mathbb{P}}[\psi(\xi, B_t^{s,\theta}, B_T^{t,\theta})|\mathcal{F}_s] = \operatorname*{ess\,sup}_{\theta\in\mathcal{A}_{s,t}^{\mathcal{U}}} \operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}_{t,T}^{\mathcal{U}}} \mathbb{E}^{\mathbb{P}}[\psi(\xi, B_t^{s,\theta}, B_T^{t,\tilde\theta})|\mathcal{F}_s].$$

Proof. It is clear that we have following equality

$$\operatorname*{ess\,sup}_{\theta\in\mathcal{A}_{s,T}^{\mathcal{U}}} \mathbb{E}^{\mathbb{P}}[\psi(\xi, B_t^{s,\theta}, B_T^{t,\theta})|\mathcal{F}_s] = \operatorname*{ess\,sup}_{\theta\in\mathcal{A}_{s,t}^{\mathcal{U}}} \operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}_{s,t,T}^{\mathcal{U}}} \mathbb{E}^{\mathbb{P}}[\psi(\xi, B_t^{s,\theta}, B_T^{t,\tilde\theta})|\mathcal{F}_s].$$

So it is sufficient to prove that

$$\operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}_{s,t,T}^{\mathcal{U}}} \mathbb{E}^{\mathbb{P}}[\psi(\xi, B_t^{s,\theta}, B_T^{t,\tilde\theta})|\mathcal{F}_s] = \operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}_{t,T}^{\mathcal{U}}} \mathbb{E}^{\mathbb{P}}[\psi(\xi, B_t^{s,\theta}, B_T^{t,\tilde\theta})|\mathcal{F}_s] \quad \mathbb{P}-a.s. \tag{3}$$

Note however that by the tower property of the conditional expectation and by the Yan's commutation theorem (see Theorem A.3 in Appendix of [9]), we can transform

LHS as follows

$$\operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}^{\mathcal{U}}_{s,t,T}} \mathbb{E}^{\mathbb{P}}[\psi(\xi, B^{s,\theta}_t, B^{t,\tilde\theta}_T)|\mathcal{F}_s] = \mathbb{E}^{\mathbb{P}}[\operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}^{\mathcal{U}}_{s,t,T}} \mathbb{E}^{\mathbb{P}}[\psi(\xi, B^{s,\theta}_t, B^{t,\tilde\theta}_T)|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}^{\mathbb{P}}[\Lambda_{s,t,T}[\zeta]|\mathcal{F}_s].$$

Here $\Lambda_{s,t,T}$ is associated with $\phi((x,y),z) := \psi(x,y,z)$ and we take $\zeta := (\xi, B^{s,\theta}_T)$. The same can be done with RHS

$$\operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}^{\mathcal{U}}_{t,T}} \mathbb{E}^{\mathbb{P}}[\psi(\xi, B^{s,\theta}_t, B^{t,\tilde\theta}_T)|\mathcal{F}_s] = \mathbb{E}^{\mathbb{P}}[\operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}^{\mathcal{U}}_{t,T}} \mathbb{E}^{\mathbb{P}}[\psi(\xi, B^{s,\theta}_t, B^{t,\tilde\theta}_T)|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}^{\mathbb{P}}[\Lambda_{t,T}[\zeta]|\mathcal{F}_s].$$

We can now easily obtain eq. (3) by applying property 3 from Lemma 12. □

Now we are ready to prove DPP.

Proof of Theorem 5. Fix $h \in [0, T-t]$. By the definition of u and the Yan's commutation theorem we have

$$\begin{aligned} u(t,x) &= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,T}} \mathbb{E}^{\mathbb{P}}[\phi(x + B^{t,\theta}_T)] = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,T}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(x + B^{t,\theta}_{t+h} + B^{t+h,\theta}_T\right)\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,T}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(x + B^{t,\theta}_{t+h} + B^{t+h,\theta}_T\right)\Big|\mathcal{F}_t\right]\right]. \end{aligned}$$

Now using the dynamic consistency from Proposition 13 (and once again Yan's commutation theorem) we get

$$\begin{aligned} u(t,x) &= \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,t+h}} \operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}^{\mathcal{U}}_{t+h,T}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(x + B^{t,\theta}_{t+h} + B^{t+h,\tilde\theta}_T\right)\Big|\mathcal{F}_t\right]\right] \\ &= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,t+h}} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}^{\mathcal{U}}_{t+h,T}} \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\phi\left(x + B^{t,\theta}_{t+h} + B^{t+h,\tilde\theta}_T\right)\Big|\mathcal{F}_{t+h}\right]\Big|\mathcal{F}_t\right]\right] \\ &= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,t+h}} \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}^{\mathcal{U}}_{t+h,T}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(x + B^{t,\theta}_{t+h} + B^{t+h,\tilde\theta}_T\right)\Big|\mathcal{F}_{t+h}\right]\Big|\mathcal{F}_t\right]\right] \\ &= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,t+h}} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}^{\mathcal{U}}_{t+h,T}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(x + B^{t,\theta}_{t+h} + B^{t+h,\tilde\theta}_T\right)\Big|\mathcal{F}_{t+h}\right]\right]. \end{aligned}$$

Applying now Lemma 11 we get that

$$u(t,x) = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,t+h}} \mathbb{E}^{\mathbb{P}}\left[\left\{\operatorname*{ess\,sup}_{\tilde\theta\in\mathcal{A}^{\mathcal{U}}_{t+h,T}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(x + y + B^{t+h,\tilde\theta}_T\right)\Big|\mathcal{F}_{t+h}\right]\right\}\Big|_{y=B^{t,\theta}_{t+h}}\right].$$

But $B_T^{t+h,\tilde{\theta}}$ is independent of $\mathcal{F}_{t+h}$. Thus

$$\begin{aligned}u(t,x) &= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,t+h}} \mathbb{E}^{\mathbb{P}}\left[\left\{\sup_{\tilde{\theta}\in\mathcal{A}^{\mathcal{U}}_{t+h,T}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(x+y+B_T^{t+h,\tilde{\theta}}\right)\right]\right\}\Big|_{y=B^{t,\theta}_{t+h}}\right] \\ &= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t,t+h}} \mathbb{E}^{\mathbb{P}}\left[u\left(t+h, x+B^{t,\theta}_{t+h}\right)\right].\end{aligned}$$

The last equality is of course by the definition of u. □

4. Capacity and related topics

Before we can define the Itô integral, we need to establish some property of the paths of a G-Lévy process with finite activity and relate different notions of regularity of random variables. In both cases, it is natural to consider the capacity framework.

4.1. Quasi-sure properties of the paths

Definition 5. *For the sublinear expectation $\mathbb{E}[.]$ with the representation*

$$\mathbb{E}[.] = \sup_{\mathbb{Q}\in\mathfrak{P}} \mathbb{E}^{\mathbb{Q}}[.]$$

we introduce the capacity related to $\mathbb{E}[.]$ as

$$c(A) = \sup_{\mathbb{Q}\in\mathfrak{P}} \mathbb{Q}(A), \quad A \in \mathcal{B}(\Omega).$$

We say that the set A is polar, if $c(A) = 0$. We say that the property holds quasi-surely (q.s.), if it holds outside the polar set.

Note that due to the representation of sublinear expectation in Corollary 8, for each G-Lévy process with finite activity on the sublinear expectation space $(\Omega, L^1_G(\Omega), \hat{\mathbb{E}})$ we can associate the family of probabilities $\mathfrak{P} = \{\mathbb{P}^\theta \colon \theta \in \mathcal{A}^{\mathcal{U}}_{0,\infty}\}$ and thus associate the capacity c, too. From now on, whenever we mention the property holding quasi-surely, it will be related to this particular capacity.

We will prove the following proposition, which enables us to work on the paths of a G-Lévy process.

Proposition 14. *Let X be a canonical process on the canonical sublinear expectation space $(\Omega, L^1_G(\Omega), \hat{\mathbb{E}})$, such that X is a G-Lévy process with finite activity under $\hat{\mathbb{E}}$. Then for each finite interval $[s,t]$ X has finite number of jumps q.s. Hence the use of the term "finite activity" is justified.*

Proof. Fix $0 \le s < t < \infty$. Define the set

$$A := \{\omega \in \Omega \colon t \mapsto X_t(\omega) \text{ has infinite number of jumps on the interval } [s,t]\}.$$

We will prove that $\mathbb{P}^\theta(A) = 0$ for each $\theta \in \mathcal{A}^{\mathcal{U}}_{0,\infty}$. Note that $\mathbb{P}^\theta = \mathbb{P} \circ (B_.^{0,\theta})^{-1}$, thus the canonical process under $\mathbb{P}^\theta$ has the same law as the integral $t \mapsto B_t^{0,\theta}$ under $\mathbb{P}$. Hence

$$\mathbb{P}^\theta(A) = \mathbb{P}(A^\theta),$$

where

$$A^\theta := \{\omega \in \Omega \colon t \mapsto B_t^{0,\theta}(\omega) \text{ has infinite number of jumps on the interval } [s,t]\}.$$

But we know that the integral has finite activity $\mathbb{P}$-a.s. as its jump part is an integral w.r.t. Poisson jump measure of the standard Poisson process. Hence $\mathbb{P}(A^\theta) = 0$. □

Remark 15. Note that of course the set A is non-empty, as there are plenty of càdlàg functions (i.e. trajectories of X), which do not exhibit finite activity. They are however negligible, as those ω's belong to a polar set.

4.2. Spaces of random variables and the relations between them

Now let us note that the we can extend our sublinear expectation to all random variables Y on Ω_T (or Ω) for which the following expression has sense

$$\hat{\mathbb{E}}[Y] := \sup_{\theta \in A^{\mathcal{U}}_{0,\infty}} \mathbb{E}^{\mathbb{P}^\theta}[Y].$$

We can thus can also extend the definition of the norm $\|.\|_p$ and define following spaces

1. Let $L^0(\Omega_T)$ be the space of all random variables on Ω_T.
2. Let $C_b(\Omega_T)$ be the space of all continuous random variables in $L^0(\Omega_T)$. The completion of $C_b(\Omega_T)$ in the norm $\|.\|_p$ will be denoted as $\mathbb{L}^p_c(\Omega_T)$.
3. Let $C_{b,lip}(\Omega_T)$ be the space of all Lipschitz continuous random variables in $C_b(\Omega_T)$. The completion of $C_{b,lip}(\Omega_T)$ in the norm $\|.\|_p$ will be denoted as $\mathbb{L}^p_{c,lip}(\Omega_T)$.

We will need the relation between these spaces and $L^p_G(\Omega_T)$ space. In the G-Brownian motion case it is well-known that $Lip(\Omega_T) \subset C_b(\Omega_T)$ and $L^p_G(\Omega_T) = \mathbb{L}^p_c(\Omega_T)$. In the case of G-Lévy process the first inclusion is untrue as the evaluations of the càdlàg paths are not continuous in the Skorohod topology (compare [2], section 'Finite-Dimensional Sets' in Chapter 3). However Ren was able to prove that $Lip(\Omega_T) \subset \mathbb{L}^1_c(\Omega_T)$ and thus $L^p_G(\Omega_T) \subset \mathbb{L}^p_c(\Omega_T)$ (see Section 5 in [11]). This relation is somehow unsatisfactory, because we would like to have a criterion for a random variable to be in our main space $L^p_G(\Omega_T)$, not the opposite. Fortunately we are able to prove that $L^p_G(\Omega_T) = \mathbb{L}^p_c(\Omega_T)$. But first let us prove with the following relation.

Proposition 16. *We have the following inclusion*

$$C_{b,lip}(\Omega_T) \subset L^1_G(\Omega_T).$$

As a consequence

$$\mathbb{L}^p_{c,lip}(\Omega_T) \subset L^p_G(\Omega_T).$$

Note that we can't use the proof from [3], which is based on the Stone-Weierstrass theorem and the tightness of the family $\{\mathbb{P}^\theta \colon \theta \in \mathcal{A}^{\mathcal{U}}_{0,T}\}$, because even though $Lip(\Omega_T)$ is an algebra which separates points in Ω_T, but it is not included either in $C_{b,lip}(\Omega_T)$ nor in $C_b(\Omega_T)$ as its elements are not continuous. Thus we need to show the proposition directly constructing an appropriate approximative sequence. We will need the following properties.

Lemma 17. *For any $\delta > 0$ and a càdlàg function $x : [0,T] \to \mathbb{R}^d$ define the following càdlàg modulus*

$$\omega'_x(\delta) := \inf_{\pi} \max_{0<i\le r} \sup_{s,t\in[t_{i-1},t_i[} |x(s) - x(t)|,$$

where infinimum runs over all partitions $\pi = \{t_0, \dots, t_r\}$ of the interval $[0,T]$ satisfying $0 = t_0 < t_1 < \ldots < t_r = T$ and $t_i - t_{i-1} > \delta$ for all $i = 1, 2, \dots, r$. Define also

$$w''_x(\delta) := \sup_{\substack{t_1\le t\le t_2 \\ t_2-t_1\le\delta}} \min\{|x(s) - x(t_1)|, |x(t_2) - x(s)|\}.$$

Then

1. $w''_x(\delta) \le w'_x(\delta)$ *for all $\delta > 0$ and $x \in \mathbb{D}_0(\mathbb{R}^+, \mathbb{R}^d)$.*
2. *For every $\epsilon > 0$ and a subinterval $[\alpha, \beta[\subset [0,T]$ if x does not have any jumps of magnitude $> \epsilon$ in the interval $[\alpha, \beta]$ then*

$$\sup_{\substack{t_1,t_2\in[\alpha,\beta[\\ |t_2-t_1|\le\delta}} |x(t_1) - x(t_2)| \le 2w''_x(\delta) + \epsilon.$$

   In particular, if x is continuous in $[\alpha, \beta[$, we have the estimate

$$\sup_{\substack{t_1,t_2\in[\alpha,\beta[\\ |t_2-t_1|\le\delta}} |x(t_1) - x(t_2)| \le 2w''_x(\delta) \le 2w'_x(\delta).$$

3. *The function $x \mapsto w'_x(\delta)$ is upper semicontinuous for all $\delta > 0$.*
4. $\lim_{\delta\downarrow 0} w'_x(\delta) \downarrow 0$ *for all $x \in \mathbb{D}_0(\mathbb{R}^+, \mathbb{R}^d)$.*

These properties are standard and might be found in [2] for properties 1, 3 and 4 (see Chapter 3, Lemma 1, eq. (14.39) and (14.46)) and [7] for property 2 (see Lemma 6.4 in Chapter VII).

Proof of Proposition 16. Fix a random variable $Y \in C_{b,lip}(\Omega_T)$. For any $n \in \mathbb{N}$ define the operator $T^n \colon \mathbb{D}_0(\mathbb{R}^+, \mathbb{R}^d) \to \mathbb{D}_0(\mathbb{R}^+, \mathbb{R}^d)$ as

$$T^n(\omega)(t) := \begin{cases} \omega_{\frac{kT}{n}} & \text{if } t \in [\frac{kT}{n}, \frac{(k+1)T}{n}[, \ k = 0, 1, \dots, n-1. \\ \omega_T & \text{if } t = T. \end{cases}$$

Define $Y^n := Y \circ T^n$. Then Y^n depend only on $\{\omega_{kT/n}\}_{k=0}^n$ thus there exists a function $\phi^n \colon \mathbb{R}^{(n+1)\times d} \to \mathbb{R}$ such that

$$Y^n(\omega) = \phi^n(\omega_0, \omega_{\frac{T}{n}}, \dots, \omega_T).$$

By the boundedness and Lipschitz continuity of Y we can easily prove that also ϕ^n must be bounded and Lipschitz continuous (all we have to do is to consider the paths, which are constant on the intervals $[kT/n, (k+1)T/n[$). Note however that

$$\hat{\mathbb{E}}[|Y - Y^n|] = \hat{\mathbb{E}}[|Y - Y \circ T^n|] \le L\, \hat{\mathbb{E}}[d(X^T, X^T \circ T^n) \wedge 2M],$$

where L and M are a Lipschitz constant and bound of Y, X^T is a canonical process, i.e. our G-Lévy process, stopped at time T and d is the Skorohod metric.

Now let us define the random variable Z^n on Ω_T as follows

$$Z^n(\omega) := \begin{cases} d(\omega, T^n(\omega)) \wedge 2M & \text{if } \omega \in A_T, \\ 0 & \text{otherwise,} \end{cases}$$

where $A_T := \{\omega \in \Omega_T \colon \omega \text{ has finite number of jumps in the interval } [0,T]\}$. By Proposition 14 we know that $Z^n = d(X^T, X^T \circ T^n) \wedge 2M$ q.s. so the expectations of both random variables are equal

$$\hat{\mathbb{E}}[d(X^T, X^T \circ T^n) \wedge 2M] = \hat{\mathbb{E}}[Z_n].$$

Thus we can only consider paths with finite number of jumps. Fix then $\omega \in A_T$ having a finite number of jumps at time $0 < r_1 < \ldots < r_{m-1} < T$ and possibly a jump at $r_m := T$. We can choose n big enough such that $r_{i+1} - r_i \geq T/n$ for $i = 0, \ldots, m-1$. Denote by A_T^n the subset of A_T containing all such ω's (i.e. with minimal distance between jumps larger or equal to T/n). We want to have an estimate of the Skorohod metric for $\omega \in A_T^n$. To obtain it we construct the piecewise linear function λ^n as follows $\lambda^n(0) = 0$, $\lambda^n(T) = T$, for each $k = 1, \ldots, n-1$ define

$$\lambda^n\left(\frac{kT}{n}\right) := \begin{cases} \frac{kT}{n} & \text{if } r_i \notin \left]\frac{(k-1)T}{n}, \frac{kT}{n}\right], \ i = 1, \ldots, m, \\ r_i & r_i \in \left]\frac{(k-1)T}{n}, \frac{kT}{n}\right], \ i = 1, \ldots, m. \end{cases}$$

Moreover, let λ^n be linear between these nods. By the construction $\|\lambda^n - Id\|_\infty \leq 2T/n$. Define $t_k := \lambda^n(kT/n)$ for $k = 0, \ldots, n$. Note that ω is continuous on $[t_k, t_{k+1}[$. Then by definition of the Skorohod metric and property 2 in Lemma 17 we have

$$\begin{aligned} d(\omega, T^n(\omega)) \wedge 2M &= \left(\inf_{\lambda \in \Lambda} \max\{\|\lambda - Id\|_\infty, \|\|T^n(\omega) - \omega \circ \lambda\|_\infty\}\right) \wedge 2M \\ &\leq (\|\lambda^n - Id\|_\infty + \|\|T^n(\omega) - \omega \circ \lambda^n\|_\infty\}) \wedge 2M \\ &\leq \left(\frac{2T}{n} + \max_{k=0,\ldots,n-1} \sup_{s,t \in [t_k, t_{k+1}[} |\omega(s) - \omega(t)|\right) \wedge 2M \\ &\leq \left[\frac{2T}{n} + 2w'_\omega\left(\frac{2T}{n}\right)\right] \wedge 2M. \end{aligned}$$

Thus we can define yet another bound M^n as

$$M^n(\omega) := \begin{cases} \left(\frac{2T}{n} + 2w'_\omega\left(\frac{2T}{n}\right)\right) \wedge 2M, & \text{if } \omega \in A_T^n, \\ 2M, & \text{if } \omega \in A_T \setminus A_T^n, \\ 2M, & \text{if } \omega \notin A_T. \end{cases}$$

Then $M^n \geq Z^n$ and thus $\hat{\mathbb{E}}[Z^n] \leq \hat{\mathbb{E}}[M^n]$. We also have $M_n \downarrow 0$ on every A_T^m. This follows from property 4 in Lemma 17. Moreover we claim than M^n is upper semi-continuous on every set A_T^m for $m \leq n$. Firstly, note that the set A_T^m is closed under the Skorohod topology. This is clear from the definition of the set: if $\{\omega^k\}_k \subset A_T^m$ then the

distance between the jumps is $\geq T/m$ for each k. But if $\omega^k \to \omega$ then also ω must satisfy this property and hence it belong to $A_T^m \subset A_T^n$. Now note that by Lemma 17, property 3, we have that $\omega \mapsto (2T/n + 2w'_\omega(2T/n)) \wedge 2M$ is upper semi-continuous as a minimum of two upper semi-continuous functions and thus

$$\begin{aligned}\limsup_{k\to\infty} M^n(\omega^k) &= \limsup_{k\to\infty} \left(\frac{2T}{n} + 2w'_{\omega^k}\left(\frac{2T}{n}\right)\right) \wedge 2M \\ &\leq \left(\frac{2T}{n} + 2w'_\omega\left(\frac{2T}{n}\right)\right) \wedge 2M = M^n(\omega).\end{aligned}$$

Thus M^n is upper semi-continuous on each closed set $A_T^m,\ m \leq n$.

We also claim that the sets A_T^m are 'big' in the sense, that the capacity of the complement is decreasing to 0. We prove it similarly to Proposition 14. Note that

$$(A_T^m)^c = \{\omega \in \Omega_T \colon \exists\, t, s \leq T,\ |t-s| < \frac{T}{m} \text{ and } \Delta\omega_t \neq 0,\ \Delta\omega_s \neq 0\}.$$

For any $\theta \in \mathcal{A}^{\mathcal{U}}_{0,T}$ define the set

$$(A_T^{m,\theta})^c = \{\omega \in \Omega_T \colon \exists\, t, s \leq T,\ |t-s| < \frac{T}{m} \text{ and } \Delta B_t^{0,\theta}(\omega) \neq 0,\ \Delta B_s^{0,\theta}(\omega) \neq 0\}.$$

Then we have then by the representation of c, the fact that $\mathbb{P}^\theta$ is the law of $B_\cdot^{0,\theta}$ (which have jumps at times when Poisson process has a jump) and the properties of the Poisson process that

$$\begin{aligned}c\left[(A_T^m)^c\right] &= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \mathbb{P}^\theta\left[(A_T^m)^c\right] = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \mathbb{P}\left[(A_T^{m,\theta})^c\right] \\ &\leq \mathbb{P}(\exists\, t, s \leq T,\ |t-s| < \frac{T}{m} \text{ and } \Delta N_t = \Delta N_s = 1) \\ &= \mathbb{P}(\exists\, s < \frac{T}{m} \text{ and } \Delta N_s = 1) = \int_0^{\frac{T}{m}} e^{-s} ds = 1 - e^{-\frac{T}{m}} \to 0.\end{aligned}$$

Note that we will prove the assertion of our proposition if we use the following lemma (proof below).

Lemma 18. *Let $\{X_n\}_n$ be a sequence of non-negative uniformly bounded random variables on Ω_T such that there exists a sequence of closed sets $(F_m)_m$ having the following properties*

1. *$c(F_m^c) \to 0$ as $m \to \infty$.*
2. *$X_n \downarrow 0$ on every F_m.*
3. *X_n is upper semi-continuous on every $F_m,\ m \leq n$.*

Then $\hat{\mathbb{E}}[X_n] \to 0$.

Applying this lemma to our sequence $\{M^n\}_n$ together with the closed sets $(A_T^m)_m$ we get that

$$\hat{\mathbb{E}}[|Y^n - Y|] \leq \hat{\mathbb{E}}[Ld(X^T, X^T \circ T^n)] \leq L\hat{\mathbb{E}}[M^n] \to 0. \qquad \square$$

Proof of Lemma 18. Fix $\epsilon > 0$. Let M be the bound of all X_n. By the representation of the sublinear expectation we have

$$\begin{aligned}
\hat{\mathbb{E}}[X_n] &= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \mathbb{E}^{\mathbb{P}^\theta}[X_n] = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \int_0^M \mathbb{P}^\theta(X_n \geq t)dt \\
&= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \int_0^M \mathbb{P}^\theta[(\{X_n \geq t\} \cap F_m) \cup (\{X_n \geq t\} \cap F_m^c)]dt \\
&\leq \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \int_0^M \mathbb{P}^\theta(\{X_n|_{F_m} \geq t\} \cup F_m^c)dt \leq \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \int_0^M [c(X_n|_{F_m} \geq t) + c(F_m^c)]\, dt \\
&\leq \int_0^M c(X_n|_{F_m} \geq t)dt + Mc(F_m^c).
\end{aligned}$$

By the first property of sets F_m we can choose m big enough so that $c(F_m^c) \leq \frac{\epsilon}{2M}$. Choose $n \geq m$. By the upper semi-continuity of X_n on F_m we get that each $\{X_n|_{F_m} \leq t\}$ is closed in the subspace topology on F_m. But F_m is also a closed set in the Skorohod topology, thus $\{X_n|_{F_m} \geq t\}$ is also closed in it. Moreover, due to monotone convergence on F_m we have that $\{X_n|_{F_m} \geq t\} \downarrow \emptyset$ as $n \uparrow \infty$. Thus by Lemma 7 in [3] we get that $c(X_n|_{F_m} \leq t) \downarrow 0$ as $n \uparrow \infty$ and we get the assertion of the lemma by applying monotone convergence theorem for the Lebesgue integral and choosing $n \geq m$ big enough, so that the integral is less then $\frac{\epsilon}{2}$. Thus

$$0 \leq \hat{\mathbb{E}}[X_n] \leq \epsilon \quad \text{for } n \text{ big enough.} \qquad \square$$

Now we are able to prove the main theorem using the reasoning by Denis *et al.* as in Theorem 52 in [3], which is based on the Stone-Weierstrass theorem.

Theorem 19. *The space $C_{b,lip}(\Omega_T)$ is dense in $C_b(\Omega_T)$ under the norm $\hat{\mathbb{E}}[|.|]$. Thus $L^1_G(\Omega_T) = \mathbb{L}^1_c(\Omega_T)$.*

Proof. Fix $Y \in C_b(\Omega_T)$. We will prove that there exists a sequence Y^n in $C_{b,lip}(\Omega_T)$ converging to Y in $\hat{\mathbb{E}}[|.|]$-norm.

Firstly, Ren proved that the family $\{P^\theta \colon \theta \in \mathcal{A}^{\mathcal{U}}_{0,T}\}$ used to represent the sublinear expectation $\hat{\mathbb{E}}[.]$ is tight (see Lemma 3.9 in [11]). Therefore by Prohorov's theorem (see e.g. Theorem 6 in [3]) for each $n \in \mathbb{N}$ there exists a set K_n which is compact in the Skorohod topology and $c(K_n^c) < 1/n$.

Note also that $C_{b,lip}(\Omega_T)$ is an subalgebra in $C_b(\Omega_T)$, which separates the points (the last claim is an easy consequence of the Tietze's extension theorem for Lipschitz functions, see Theorem 1.5.6. in [13]). Thus by the Stone-Weierstrass theorem for each compact set K_n there exists a random random variable Z^n on K_n, which is bounded and Lipschitz and

$$\sup_{\omega\in K_n} |Y(\omega) - Z^n(\omega)| < \frac{1}{n} \quad \text{and} \quad \sup_{\omega\in K_n} |Z^n(\omega)| \leq \sup_{\omega\in K_n} |Y(\omega)|.$$

Once again using the Tietze's extension theorem, we may extend Z^n to the whole Ω_T preserving the Lipschitz constant and the bound and we will denote this extension by

Y^n. Note that $Y^n \in C_{b,lip}(\Omega_T)$ and that

$$\sup_{\omega\in\Omega_T} |Y^n(\omega)| = \sup_{\omega\in K_n} |Z^n(\omega)| \leq \sup_{\omega\in K_n} |Y(\omega)| \leq \sup_{\omega\in\Omega_T} |Y(\omega)| =: M.$$

Hence

$$\begin{aligned}\hat{\mathbb{E}}[|Y^n - Y|] &\leq \hat{\mathbb{E}}[|Y^n - Y|\mathbb{1}_{K_n}] + \hat{\mathbb{E}}[|Y^n - Y|\mathbb{1}_{K_n^c}] \leq 2Mc(K_n^c) + \frac{1}{n}c(K_n)\\ &\leq \frac{1}{n}(2M+1) \to 0.\end{aligned}$$

Therefore we proved that $C_{b,lip}(\Omega_T)$ is dense in $C_b(\Omega_T)$. Thus the closure of $C_{b,lip}(\Omega_T)$ under the norm $\hat{\mathbb{E}}[|.|]$ is exactly $\mathbb{L}_c^1(\Omega_T)$. However, by Proposition 16 we know that $C_{b,lip}(\Omega_T) \subset L_G^1(\Omega_T)$ and by Remark 5 in [11] we know that $L_G^1(\Omega_T) \subset \mathbb{L}_c^1(\Omega_T)$. Thus $L_G^1(\Omega_T) = \mathbb{L}_c^1(\Omega_T)$. □

The theorem above allows us to use the characterization of the random variables in $\mathbb{L}_c^p(\Omega)$ in terms of their continuity and thickness of their tails. Namely, introduce the following definition.

Definition 6. *We will say that the random variable $Y \in L^0(\Omega)$ is quasi-continuous, if for all $\epsilon > 0$ there exists an open set O such that $c(O) < \epsilon$ and $Y|_{O^c}$ is continuous. For convenience, we will often use the abbreviation* q.c.

It is well known that the following characterization of $\mathbb{L}_c^p(\Omega)$ (thus also $L_G^p(\Omega)$) holds (see Theorem 25 in [3]).

Proposition 20. *For each $p \geq 1$ one has*

$$\mathbb{L}_c^p(\Omega) = L_G^p(\Omega) = \{Y \in L^0(\Omega)\colon \lim_{n\to\infty} \hat{\mathbb{E}}[|Y|^p \mathbb{1}_{\{|Y|>n\}}] = 0,\ Y \textit{ has a q.c. version}\}.$$

Thanks to this proposition we will be sure that our integral really belong to the $L_G^2(\Omega)$ space.

5. Definition of the Itô integral

By the definition of the G-Lévy process there is a Lévy -Itô -type decomposition on the continuous part (i.e. generalized G-Brownian motion) and pure-jump process. As it is widely known, there is a good definition of the Itô integral w.r.t. G-Brownian motion, so let us assume that we have to deal only with a pure-jump G-Lévy process X, i.e. $\mathcal{U} = \mathcal{V} \times \{0\} \times \{0\}$.

We introduce the following random measure: for any $0 \leq t < s$ and $A \in \mathcal{B}(\mathbb{R}^d)$

$$X(]t,s], A) := \sum_{t<u\leq s} \mathbb{1}_A(\Delta X_u),\ q.s.$$

Remark 21. Note that this random measure is well-defined q.s. thanks to the finite activity property. Moreover, it is really a random measure, i.e. is countably additive. This is not true if one would like to compensate it by factor $\hat{\mathbb{E}}[X(]t,s],A)]$, as a a function

$A \to \hat{\mathbb{E}}[X(]t,s],A)]$ is usually not additive, as one can easily check. Though it is not such a big obstacle for defining the Itô integral, it shows a big difference to the ordinary Lévy jump measures, which can always be compensated (compare [1]). Moreover, it shows that for disjoints sets A_1 and A_2 the random variables $X(]t,s],A_1)$ and $X(]t,s],A_2)$ are NOT independent under $\hat{\mathbb{E}}[.]$[2]!

Let us now introduce the set of simple integrands.

Definition 7. *Let $\mathcal{H}_G^S([0,T]\times\mathbb{R}^d)$ be a collection of all processes defined on $[0,T]\times\mathbb{R}^d\times\Omega$ of the form*

$$K(u,z)(\omega) = \sum_{k=1}^{n-1}\sum_{l=1}^{m} F_{k,l}(\omega)\,\mathbb{1}_{]t_k,t_{k+1}]}(u)\,\psi_l(z),\ \ n,m\in\mathbb{N}, \tag{4}$$

where $0\leq t_1<\ldots<t_n\leq T$ is the partition of $[0,T]$, $\{\psi_l\}_{l=1}^m\subset C_{b,lip}(\mathbb{R}^d)$ are functions with disjoint supports s.t. $\psi_l(0)=0$ and $F_{k,l}=\phi_{k,l}(X_{t_1},\ldots,X_{t_k}-X_{t_{k-1}})$, $\phi_{k,l}\in C_{b,lip}(\mathbb{R}^{d\times k})$. We equip this space with the following norm

$$\|K\|_{\mathcal{H}_G^2([0,T]\times\mathbb{R}^d)} := \hat{\mathbb{E}}\left[\int_0^T \sup_{v\in\mathcal{V}}\int_{\mathbb{R}^d} K^2(u,z)v(dz)du\right].$$

(This is of course the norm up to equivalence classes).

Definition 8. *Let $0\leq s<t\leq T$. The Itô integral of $K\in\mathcal{H}_G^S([0,T]\times\mathbb{R}^d)$ w.r.t. jump measure X is a random variable defined as*

$$\int_s^t\int_{\mathbb{R}^d} K(u,z)X(du,dz) := \sum_{s<u\leq t} K(u,\Delta X_u),\ \ q.s.$$

If $s=0,\ t=T$ we will denote also the integral as an operator I.

Theorem 22. *For every $K\in\mathcal{H}_G^S([0,T]\times\mathbb{R}^d)$ we have that*

$$\int_s^t\int_{\mathbb{R}^d} K(u,z)X(du,dz)\in L_G^2(\Omega).$$

Proof. By linearity of $L_G^2(\Omega)$ it suffices to prove the assertion of the theorem for K of the form

$$K(u,z) := \psi(z), \quad \psi\in C_{b,lip}(\mathbb{R}^d).$$

Note that we can take K deterministic, because for every $X\in L_G^2(\Omega)$ and $\xi\in Lip(\Omega)$, $X\cdot\xi\in L_G^2(\Omega)$ due to the boundedness of ξ. Thus it suffices to prove that

$$\sum_{s<u\leq t}\psi(\Delta X_u) =: Y\in L_G^2(\Omega).$$

[2] This is not surprising, as there is already a good characterization of the random variables mutually independent of each other under sublinear expectations, see [5].

We will use Proposition 20. Firstly, we will prove that Y has a quasi-continuous version (or rather quasi-Lipschitz-continuous). Introduce a random variable

$$Z := \sum_{s \le u \le t} \psi(\Delta X_u)$$

and a set

$$A_n := \{\omega \in \Omega \colon \omega \text{ has at most } n \text{ jumps in the interval }]s,t[\\ \text{ and no jumps in both }]s-1/n, s[\text{ and }]t, t+1/n[\,\}.$$

Fix a sequence $(\omega^m)_m \subset A_n$ converging to ω in the Skorohod topology. By the definition of the Skorohod metric it is easy to see that if $\Delta\omega_u \neq 0$, $u \in]s,t[$, then there exists a sequence $(u_m)_m \subset]s,t[$ s.t. $\Delta\omega^m_{u_m} \to \Delta\omega_u$. Conversely, if there exists a sequence $(u_m)_m \subset]s,t[$ converging to $u \in]s,t[$ and such that $\Delta\omega^m_{u_m} \neq 0$ for almost all m, then $\Delta\omega^m_{u_m} \to \Delta\omega_u$ (which might be equal to 0). By this we see that ω has at most n jumps in the interval $]s,t[$. Similarly, we conclude that ω can't have any jumps in the intervals $]s-1/n, s[$ and $]t, t+1/n[$. Thus, A_n is a closed set under Skorohod topology.

Note that we have the estimate

$$|\Delta\omega^m_{u_m} - \Delta\omega_u| \le 2d(\omega, \omega^m).$$

Moreover, by the definition of A_n we have that jumps of ω^m can neither escape the interval $[s,t]$ nor enter it. Thus

$$\sum_{s \le u \le t} \psi(\Delta\omega^m_u) \to \sum_{s \le u \le t} \psi(\Delta\omega_u)$$

or to be more exact

$$\left| \sum_{s \le u \le t} \psi(\Delta\omega^m_u) - \sum_{s \le u \le t} \psi(\Delta\omega_u) \right| \le 2L(n+2) \cdot d(\omega, \omega_m),$$

where L is the Lipschitz constant of ψ. Hence, Z is Lipschitz continuous on A_n.

Will prove now that the complement of A_n has a small capacity. By the same argument as in the Proposition 14, we can show that X has at least n jumps in the interval $]\alpha, \beta[$ implies that the Poisson process needs to have also at least n jumps in the same interval and the capacity of set A_n^c might be dominated in the following manner

$$\begin{aligned} c(A_n^c) \le & \mathbb{P}(N \text{ has at least } n \text{ jumps in the interval }]s,t[) \\ & + \mathbb{P}(N \text{ has at least 1 jump in the interval }]s-1/n, s[) \\ & + \mathbb{P}(N \text{ has at least 1 jump in the interval }]t, t+1/n[) \downarrow 0. \end{aligned}$$

Moreover, A_n^c is open as the complement of a closed set. Hence, we conclude that Z is quasi-continuous.

It remains to show that $Z = Y$ q.s. This is however easy, since

$$c(Y \neq Z) = c(\Delta X_s \neq 0) \le \mathbb{P}(\Delta N_s \neq 0) = 0.$$

The 'uniform integrity condition' might be proved in the similar manner. Let M be a bound of ψ. Without the loss of generality we may assume that $M = 1$. Note that for any $\theta \in \mathcal{A}^{\mathcal{U}}_{0,\infty}$ the following set

$$\left\{ \left| \sum_{s<u\le t} \psi(\theta_u \Delta N_u) \right| > n \right\} \subset \{N \text{ has at least } n \text{ jumps in the interval }]s,t]\} =: B_n,$$

as the sum of jumps grows only at jump times and only by a value bounded by 1. Introduce

$$C_n := B_n \setminus B_{n+1} = \{N \text{ has } n \text{ jumps in the interval }]s,t]\}$$

Hence we have the estimate

$$\begin{aligned}
\hat{\mathbb{E}}\left[|Y|^2 \mathbb{1}_{\{|Y|>n\}}\right] &= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \mathbb{E}^{\mathbb{P}}\left[\left| \sum_{s<u\le t} \psi(\Delta B^{0,\theta}_u) \right|^2 \mathbb{1}_{\left\{\left|\sum_{s<u\le t}\psi(\Delta B^{0,\theta}_u)\right|>n\right\}} \right] \\
&= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \mathbb{E}^{\mathbb{P}}\left[\left| \sum_{s<u\le t} \psi(\theta_u \Delta N_u) \right|^2 \mathbb{1}_{\left\{\left|\sum_{s<u\le t}\psi(\theta_u \Delta N_u)\right|>n\right\}} \right] \\
&\le \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \mathbb{E}^{\mathbb{P}}\left[\sum_{s<u\le t} |\psi(\theta_u \Delta N_u)|^2 \mathbb{1}_{B_n} \right] \le \sum_{m=n}^{\infty} \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \mathbb{E}^{\mathbb{P}}\left[\sum_{s<u\le t} |\psi(\theta_u \Delta N_u)|^2 \mathbb{1}_{C_m} \right] \\
&\le \sum_{m=n}^{\infty} \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \mathbb{E}^{\mathbb{P}}\left[m^2 \mathbb{1}_{C_m}\right] = \sum_{m=n}^{\infty} m^2 \mathbb{P}(C_m), \to 0 \text{ as } n \to \infty,
\end{aligned}$$

because the Poisson random variable has second moment finite and the number of jumps of the Poisson process in the fixed intervals is Poisson-distributed.

We conclude the proof by noting that Y satisfies the characterization in Proposition 20, thus belongs to the space $L^2_G(\Omega)$. □

Theorem 23. *Itô integral I is a continuous linear operator from $\mathcal{H}^S_G([0,T]\times\mathbb{R}^d)$ to $L^2_G(\Omega_T)$.*

Proof. We will utilize Corollary 8. Namely, let K has representation as in eq. (4), i.e.

$$K(u,z)(\omega) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} F_{k,l}(\omega)\, \mathbb{1}_{]t_k,t_{k+1}]}(u)\, \psi_l(z)$$

for some partition $\pi = \{t_1,\dots,t_n\}$ of the interval $[0,T]$, bounded Lipschitz continuous functions $\{\psi_l\}_{l=1}^m$ with disjoint supports such that $\psi_l(0) = 0$ and random variables $F_{k,l} = \phi_{k,l}(X_{t_1},\dots,X_{t_k} - X_{t_{k-1}})$, $\phi_{k,l} \in C_{b,lip}(\mathbb{R}^{d\times k})$. By the corollary and the definition of the Itô integral we have

$$\hat{\mathbb{E}}\left[\left(\int_0^T\int_{\mathbb{R}^d}K(u,z)X(du,dz)\right)^2\right]=\sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}}\mathbb{E}^{\mathbb{P}^\theta}\left[\left(\sum_{0<u\leq T}K(u,\Delta X_u)\right)^2\right]$$

$$=\sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}}\mathbb{E}^{\mathbb{P}^\theta}\left[\left(\sum_{0<u\leq T}\sum_{k=1}^{n-1}\sum_{l=1}^{m}\phi_{k,l}(X_{t_1},\dots,X_{t_k}-X_{t_{k-1}})\,\mathbb{1}_{]t_k,t_{k+1}]}(u)\,\psi_l(\Delta X_u)\right)^2\right]$$

$$=\sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}}\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{0<u\leq T}\sum_{k=1}^{n-1}\sum_{l=1}^{m}\phi_{k,l}(B_{t_1}^{0,\theta},\dots,B_{t_k}^{t_{k-1},\theta})\,\mathbb{1}_{]t_k,t_{k+1}]}(u)\,\psi_l(\Delta B_u^{0,\theta})\right)^2\right]$$

$$=\sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}}\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{0<u\leq T}\sum_{k=1}^{n-1}\sum_{l=1}^{m}\phi_{k,l}(B_{t_1}^{0,\theta},\dots,B_{t_k}^{t_{k-1},\theta})\,\mathbb{1}_{]t_k,t_{k+1}]}(u)\,\psi_l(\theta_u\Delta N_u)\right)^2\right]. \quad (5)$$

Define a predictable process $K^\theta(u,z)$ as

$$K^\theta(u,z):=\sum_{k=1}^{n-1}\sum_{l=1}^{m}\phi_{k,l}(B_{t_1}^{0,\theta},\dots,B_{t_k}^{t_{k-1},\theta})\,\mathbb{1}_{]t_k,t_{k+1}]}(u)\,\psi_l(\theta_u\,z).$$

Then we can write eq. (5) as

$$\hat{\mathbb{E}}\left[\left(\int_0^T\int_{\mathbb{R}^d}K(u,z)X(du,dz)\right)^2\right]=\sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\int_{\mathbb{R}^d}K^\theta(u,z)N(du,dz)\right)^2\right],$$

$$=\sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\int_{\mathbb{R}^d}K^\theta(u,z)\tilde{N}(du,dz)+\int_0^T K^\theta(u,1)du\right)^2\right],$$

where $N(du,dz)$ and $\tilde{N}(du,dz)$ are respectively the Poisson random measure associated with the standard Poisson process and the compensated Poisson measure. Using standard inequalities we get hence:

$$\hat{\mathbb{E}}\left[\left(\int_0^T\int_{\mathbb{R}^d}K(u,z)X(du,dz)\right)^2\right]$$

$$\leq 2\sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}}\left\{\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T\int_{\mathbb{R}^d}K^\theta(u,z)\tilde{N}(du,dz)\right)^2\right]+\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T K^\theta(u,1)du\right)^2\right]\right\}$$

$$\leq 2\sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}}\left\{\int_0^T\mathbb{E}^{\mathbb{P}}\left[\left(K^\theta(u,1)\right)^2\right]du+T\int_0^T\mathbb{E}^{\mathbb{P}}\left[\left(K^\theta(u,1)\right)^2\right]du\right\}$$

$$=2(T+1)\sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}}\int_0^T\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{k=1}^{n-1}\sum_{l=1}^{m}\phi_{k,l}(B_{t_1}^{0,\theta},\dots,B_{t_k}^{t_{k-1},\theta})\,\mathbb{1}_{]t_k,t_{k+1}]}(u)\,\psi_l(\theta_u)\right)^2\right]du$$

Let $C_T := 2(T+1)$. Note that the intervals $]t_k, t_{k+1}]$ are mutually disjoint, just as the supports of ψ_l, hence

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left(\int_0^T \int_{\mathbb{R}^d} K(u,z)X(du,dz)\right)^2\right] \\
&\leq C_T \sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}} \sum_{k=1}^{n-1}\sum_{l=1}^{m} \int_0^T \mathbb{E}^{\mathbb{P}}\left[\phi_{k,l}^2(B_{t_1}^{0,\theta},\ldots,B_{t_k}^{t_{k-1},\theta})\,\mathbb{1}_{]t_k,t_{k+1}]}(u)\,\psi_l^2(\theta_u)\right] du \\
&= C_T \sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}} \sum_{k=1}^{n-1}\sum_{l=1}^{m} \int_{t_k}^{t_{k+1}} \mathbb{E}^{\mathbb{P}}\left[\phi_{k,l}^2(B_{t_1}^{0,\theta},\ldots,B_{t_k}^{t_{k-1},\theta})\,\psi_l^2(\theta_u)\right] du. \qquad (6)
\end{aligned}$$

Note now that the process $(\theta_s)_{s\in]t_1,t_k]}$ is independent of the process $(\theta_s)_{s\in]t_k,t_{k+1}]}$ due to the choice of filtration. As a consequence the expectation in eq. (6) factorizes and we have

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\left(\int_0^T \int_{\mathbb{R}^d} K(u,z)X(du,dz)\right)^2\right] \\
&\leq C_T \sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}} \sum_{k=1}^{n-1}\sum_{l=1}^{m} \int_{t_k}^{t_{k+1}} \mathbb{E}^{\mathbb{P}}\left[\phi_{k,l}^2(B_{t_1}^{0,\theta},\ldots,B_{t_k}^{t_{k-1},\theta})\right]\,\mathbb{E}^{\mathbb{P}}\left[\psi_l^2(\theta_u)\right] du \\
&= C_T \sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}} \sum_{k=1}^{n-1}\sum_{l=1}^{m} \int_{t_k}^{t_{k+1}} \mathbb{E}^{\mathbb{P}}\left[\phi_{k,l}^2(B_{t_1}^{0,\theta},\ldots,B_{t_k}^{t_{k-1},\theta})\right] \int_{\mathbb{R}^d} \psi_l^2(z)\mu_{\theta_u}(dz)du \\
&= C_T \sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_{\mathbb{R}^d} \sum_{k=1}^{n-1}\sum_{l=1}^{m} \phi_{k,l}^2(B_{t_1}^{0,\theta},\ldots,B_{t_k}^{t_{k-1},\theta})\,\mathbb{1}_{]t_k,t_{k+1}]}(u)\,\psi_l^2(z)\mu_{\theta_u}(dz)du\right] \\
&\leq C_T \sup_{\theta\in\mathcal{A}_\pi^{\mathcal{U}}} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \sup_{v\in\mathcal{V}} \int_{\mathbb{R}^d} \sum_{k=1}^{n-1}\sum_{l=1}^{m} \phi_{k,l}^2(B_{t_1}^{0,\theta},\ldots,B_{t_k}^{t_{k-1},\theta})\,\mathbb{1}_{]t_k,t_{k+1}]}(u)\,\psi_l^2(z)\,v(dz)du\right] \\
&= C_T\,\hat{\mathbb{E}}\left[\int_0^T \sup_{v\in\mathcal{V}} \int_{\mathbb{R}^d} K^2(u,z)\,v(dz)du\right].
\end{aligned}$$

□

Corollary 24. *Let $\mathcal{H}_G^2([0,T]\times\mathbb{R}^d)$ denote the topological completion of $\mathcal{H}_G^S([0,T]\times\mathbb{R}^d)$ under the norm $\|.\|_{\mathcal{H}_G^2([0,T]\times\mathbb{R}^d)}$. Then Itô integral can be extended by the continuity of the operator I to the whole space $\mathcal{H}_G^2([0,T]\times\mathbb{R}^d)$. Moreover, the formula from Definition 8 still holds for all $K\in\mathcal{H}_G^2([0,T]\times\mathbb{R}^d)$.*

Remark 25. Since the jump measure is not compensated, there is no reason to expect that the expectation of such an integral should be 0. Moreover, it is easy to check that in general the Itô-Lévy integral is not a symmetric random variable. As the consequence the nature of this integral is significantly different from Itô integral w.r.t. G-Brownian motion.

Remark 26. We may compute explicitly the expectation of the integral in terms of the integral w.r.t. Lévy measure. Namely, for any $\xi \in L^1_G(\Omega)$ and $K \in \mathcal{H}^2_G([0,T]\times\mathbb{R}^d)$ we have

$$\hat{\mathbb{E}}\left[\xi + \int_0^T \int_{\mathbb{R}^d} K(u,z)X(du,dz)\right] = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{\pi}} \mathbb{E}^{\mathbb{P}^\theta}\left[\xi + \int_0^T \int_{\mathbb{R}^d} K(u,z)\mu_{\theta_u}(dz)du\right],$$

where π is any finite partition of the interval $[0,\infty[$. The technique for proving this is exactly the same as for Theroem 23. We need to prove it only for $K \in H^S_G([0,T]\times\mathbb{R}^d)$ and by using the representation formula for sublinear expectation (as above), we may get the desired equality. (Note that we don't need to use any inequality, as it was necessary in Theroem 23, thus we really get an equality).

6. *G*-Itô-Lévy processes. Itô formula

In this section we will introduce G-Itô Lévy processes. Assume $\mathcal{U}$ to be of the form $\mathcal{U} := \mathcal{V}\times\{0\}\times\mathcal{Q}$, i.e. there is no drift-uncertainty. Assume moreover that the set $\mathcal{Q}$ is bounded and convex. Then the G-Lévy process X associated with $\mathcal{U}$ might be represented as $X := B + L$, where B is a G-Brownian motion associated with $\mathcal{Q}$ and L is a pure-jump G-Lévy process associated with $\mathcal{V}$.[3]

Recall the following notation

$$x\cdot y := x^T y, \quad |x| := \sqrt{x\cdot x}, \quad \text{and } \gamma\colon\beta := \operatorname{tr}(\gamma\beta), \quad |\gamma| := \sqrt{\gamma\colon\gamma},$$

where $x, y, \in \mathbb{R}^d$, $\gamma, \beta \in \mathbb{S}^d$ ($\mathbb{S}^d$ is the space of all $d\times d$-dimensional symmetric matrices).

We will also use the following definition: the process Z taking values in a metric space $(\mathcal{X}, d)$ is an elementary process, if it has the form

$$Z_t = \sum_{n=1}^N \phi_n(X_{t_1},\dots,X_{t_n})\mathbb{1}_{]t_{n-1},t_n]},$$

where $0 \le t_1 < \dots t_N < \infty$ and $\phi_n\colon \mathbb{R}^{d\times n} \to \mathcal{X}$ is Lipschitz continuous and bounded.

Define the following spaces

1. Let $\mathcal{H}^2_G(0,T)$ denote the completion of all $\mathbb{R}^d$-valued elementary processes under the norm
$$\|Z\|^2_{\mathcal{H}^2_G(0,T)} := \hat{\mathbb{E}}\left[\int_0^T Z_s Z_s^T\colon d\langle B\rangle_s\right].$$
For a process $Z \in \mathcal{H}^2_G(0,T)$ one can define the stochastic integral denoted by $\int_0^t Z_s\cdot dB_s$. Note that usually one uses different elementary processes (with random variables which are cylinder functions of the G-Brownian motion and not a G-Lévy process). However we can easily generalize the Itô integral for this larger class of

[3]Formally, we should introduce new operators G^c and G^d which would produce G^c-Brownian motion and pure jump G^d-Lévy process. However, we think that in this paper this slight abuse of notation does not lead to any confusion, so we will keep it.

integrands, because the increment of the G-Brownian motion is independent of the past of the G-Lévy process. [4]

2. Let $M_G^1(0,T)$ denote the completion of all $\mathbb{R}$-valued elementary processes under the norm
$$\|\eta\|_{M_G^1(0,T)} := \hat{\mathbb{E}}\left[\int_0^T |\eta_s| ds\right].$$
For a process $\eta \in M_G^1(0,T)$ one can define the following integral $\int_0^t \eta_s ds$.
3. Let $\mathcal{M}_G^1(0,T)$ denote the completion of all $\mathbb{S}^d$-valued elementary processes under the norm
$$\|\eta\|_{\mathcal{M}_G^1(0,T)} := \hat{\mathbb{E}}\left[\int_0^T |\eta_s| ds\right].$$
For a process $\eta \in \mathcal{M}_G^1(0,T)$ one can define the following integral $\int_0^t \eta_s \colon d\langle B\rangle_s$.

Definition 9. *The process $Y = (Y^1, \dots, Y^m)$ is called a G-Itô-Lévy process in $\mathbb{R}^m$, if there exists processes $Z^i \in \mathcal{H}_G^2(0,T)$, $\alpha^i \in M_G^1(0,T)$, $\beta^i \in \mathcal{M}_G^1(0,T)$ and $K^i \in \mathcal{H}_G^2([0,T] \times \mathbb{R}^d)$, $i = 1, \dots, m$ such that for all $t \in [0,T]$*
$$Y_t^i = Y_0^i + \int_0^t \alpha_s^i ds + \int_0^t \beta_s^i : d\langle B\rangle_s + \int_0^t Z_s^i \cdot dB_s + \int_0^t \int_{\mathbb{R}^n} K^i(s,z) L(ds,dz). \ q.s. \quad (7)$$

Theorem 27 (Itô formula)**.** *Let Y be a G-Itô-Lévy process in $\mathbb{R}^m$ with representation (7). Let $f \in C^2(\mathbb{R}^m)$. Then $f(Y_t)$ is also a G-Lévy -Itô process with the representation*
$$\begin{aligned} f(Y_t) &= f(Y_0) + \sum_{i=1}^m \int_0^t \frac{\partial f}{\partial x_i}(Y_s)\alpha_s^i ds + \sum_{i=1}^m \int_0^t \frac{\partial f}{\partial x_i}(Y_s)\beta_s^i \colon d\langle B\rangle_s \\ &+ \frac{1}{2}\sum_{i,j=1}^m \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_s) Z_s^i (Z_s^j)^T \colon d\langle B\rangle_s + \sum_{i=1}^m \int_0^t \frac{\partial f}{\partial x_i}(Y_s) Z_s^i \cdot dB_s \\ &+ \int_0^t \int_{\mathbb{R}^d} \left[f(Y_{s-} + K(s,z)) - f(Y_{s-})\right] L(ds,dz), q.s. \end{aligned}$$
where $K := (K^1, \dots, K^m)$.

Proof. Firstly, define the following random times
$$\tau_0 = 0, \quad \tau_n := \inf\{t > \tau_{n-1} \colon 0 \neq \Delta X_t (= \Delta L_t)\}, \ n = 1, 2, \dots$$
Each τ_n is a stopping time w.r.t. filtration generated by the canonical process X. Due to finite activity $\tau_n \uparrow \infty$ q.s. Thus we have
$$\begin{aligned} f(Y_t) - f(Y_0) &= \sum_{n=1}^\infty \left[f(Y_{t\wedge\tau_n}) - f(Y_{t\wedge\tau_{n-1}})\right] \\ &= \sum_{n=1}^\infty \left[f(Y_{t\wedge\tau_n -}) - f(Y_{t\wedge\tau_{n-1}})\right] + \sum_{n=1}^\infty \left[f(Y_{t\wedge\tau_n}) - f(Y_{t\wedge\tau_n -})\right], \ q.s. \quad (8) \end{aligned}$$

[4] It is easy to see that if $\mathcal{Q}$ is not only bounded but also bounded away from 0, then there exist constants $0 < A \leq B$ such that $At \cdot Id_d \leq \langle B\rangle_t \leq Bt \cdot Id_d$ and the norm $\mathcal{H}_G^2(0,T)$ is equivalent to the following norm: $\hat{\mathbb{E}}\left[\int_0^T |Z_s|^2 ds\right]^{1/2}$.

Note that the second sum might be written as

$$\begin{aligned}\sum_{n=1}^{\infty}[f(Y_{t\wedge\tau_n})-f(Y_{t\wedge\tau_n-})] &= \sum_{n=1}^{\infty}[f(Y_{t\wedge\tau_n-}+K(t\wedge\tau_n,\Delta L_{t\wedge\tau_n})-f(Y_{t\wedge\tau_n-})]\\ &= \int_0^t\int_{\mathbb{R}^d}[f(Y_{s-}+K(s,z))-f(Y_{s-})]\,L(ds,dz),\ q.s. \qquad (9)\end{aligned}$$

The first sum is more complicated and one has to be cautious by dealing with stopping times here. Fix n and introduce the process $Y_{n,t}:=(Y^1_{n,t},\ldots,Y^m_{n,t})$ where $Y^i_{n,t}$ is defined as

$$Y^i_{n,t}=Y^i_{t\wedge\tau_{n-1}}+\int_0^t\alpha^i_s\mathbb{1}_{]\tau_{n-1},\tau_n]}ds+\int_0^t\beta^i_s\mathbb{1}_{]\tau_{n-1},\tau_n]}:d\langle B\rangle_s+\int_0^t Z^i_s\mathbb{1}_{]\tau_{n-1},\tau_n]}\cdot dB_s.$$

Note that the integrands may fall out of their spaces, when multiplied by a factor $\mathbb{1}_{]\tau_{n-1},\tau_n]}$, as the multiplied integrands might lose so-called quasi-continuity (see [3] and [12] for the discussion of this problem). This general problem is not an obstacle for us, if we use the definition of integrals which does not assume the quasi-continuity of the integrand. Such a definition was introduced by Li and Peng in [6] and we can utilize it immediately. They also gave the Itô formula for such processes (Theorem 5.4), which we apply now to the process Y^n_t and f. Thus

$$\begin{aligned}f(Y_{n,t}) =& f(Y_{t\wedge\tau_{n-1}})+\sum_{i=1}^m\int_0^t\frac{\partial f}{\partial x_i}(Y_{n,s})\alpha^i_s\mathbb{1}_{]\tau_{n-1},\tau_n]}ds\\ &+\sum_{i=1}^m\int_0^t\frac{\partial f}{\partial x_i}(Y_{n,s})\beta^i_s\mathbb{1}_{]\tau_{n-1},\tau_n]}:d\langle B\rangle_s+\sum_{i=1}^m\int_0^t\frac{\partial f}{\partial x_i}(Y_{n,s})Z^i_s\mathbb{1}_{]\tau_{n-1},\tau_n]}\cdot dB_s\\ &+\frac12\sum_{i,j=1}^m\int_0^t\frac{\partial^2 f}{\partial x_i\partial x_j}(Y_{n,s})Z^i_s(Z^j_s)^T\mathbb{1}_{]\tau_{n-1},\tau_n]}:d\langle B\rangle_s,\ q.s. \qquad (10)\end{aligned}$$

Firstly, notice that by Lemma 4.3 in [6] one has

$$Y^i_{n,t}=Y^i_{t\wedge\tau_{n-1}}+\int_{]t\wedge\tau_{n-1},t\wedge\tau_n]}\left[\alpha^i_s ds+\beta^i_s:d\langle B\rangle_s+Z^i_s\cdot dB_s\right],\quad q.s.$$

Thus $Y^i_{n,t}=Y^i_t$ q.s. on $[\tau_{n-1},\tau_n[$, $i=1,\ldots,m$ and hence we can rewrite (10) as

$$\begin{aligned}f(Y_{t\wedge\tau_n-}) =& f(Y_{t\wedge\tau_{n-1}})+\sum_{i=1}^m\int_0^t\frac{\partial f}{\partial x_i}(Y_s)\alpha^i_s\mathbb{1}_{]\tau_{n-1},\tau_n]}ds\\ &+\sum_{i=1}^m\int_0^t\frac{\partial f}{\partial x_i}(Y_s)\beta^i_s\mathbb{1}_{]\tau_{n-1},\tau_n]}:d\langle B\rangle_s+\sum_{i=1}^m\int_0^t\frac{\partial f}{\partial x_i}(Y_s)Z^i_s\mathbb{1}_{]\tau_{n-1},\tau_n]}\cdot dB_s\\ &+\frac12\sum_{i,j=1}^m\int_0^t\frac{\partial^2 f}{\partial x_i\partial x_j}(Y_s)Z^i_s(Z^j_s)^T\mathbb{1}_{]\tau_{n-1},\tau_n]}:d\langle B\rangle_s,\ q.s. \qquad (11)\end{aligned}$$

Now taking a sum in (11) we get that

$$\sum_{n=1}^{\infty}\left[f(Y_{t\wedge\tau_n-})-f(Y_{t\wedge\tau_{n-1}})\right]=\sum_{i=1}^{m}\int_0^t\frac{\partial f}{\partial x_i}(Y_s)\alpha_s^i ds+\sum_{i=1}^{m}\int_0^t\frac{\partial f}{\partial x_i}(Y_s)\beta_s^i\colon d\langle B\rangle_s$$
$$+\frac{1}{2}\sum_{i,j=1}^{m}\int_0^t\frac{\partial^2 f}{\partial x_i\partial x_j}(Y_s)Z_s^i(Z_s^j)^T\colon d\langle B\rangle_s+\sum_{i=1}^{m}\int_0^t\frac{\partial f}{\partial x_i}(Y_s)Z_s^i\cdot dB_s,\ q.s. \quad (12)$$

Combining eq. (8), (9) and (12) we get the assertion of the theorem. □

7. Diffusions with jump uncertainty

At the end of this section, we will establish the SDE's and BSDE's w.r.t. G-Lévy processes. Once again we assume that $\mathcal{U}=\mathcal{V}\times\{0\}\times\mathcal{Q}$ with $\mathcal{Q}$ bounded and convex. Thus the quadratic variation $\langle B\rangle_t$ might be dominated by $M\cdot t\cdot Id_d$ for some constant M, or more specifically, the quadratic covariation $\langle B^i,B^j\rangle_t$ might be dominated by $M^{i,j}\cdot t$.

We will follow the idea presented by Peng in [10], Chapter V.

Let us introduce the new norm on the integrands: for a $\mathbb{R}^n$-dimensional process Z define

$$\|Z\|^p_{\hat{\mathcal{M}}^p_G(0,T)}:=\int_0^T\hat{\mathbb{E}}[|Z_t|^p]dt,\ p\geq 1.$$

The completion of the space of n-dimensional elementary processes under this norm will be denoted as $\hat{\mathcal{M}}^p_G(0,T)$. Note that

$$\hat{\mathbb{E}}\left[\int_0^T|Z_t|^p dt\right]\leq\int_0^T\hat{\mathbb{E}}[|Z_t|^p]dt,$$

thus appropriate integrals will be always well defined.

Similarly, we need to adjust the space of integrands for the jump measure. Let $\hat{\mathcal{H}}^2_G([0,T]\times\mathbb{R}^d)$ denote the completion of all $\mathcal{H}^S_G([0,T]\times\mathbb{R}^d)$ under the norm

$$\|K\|^2_{\hat{\mathcal{H}}^2_G([0,T]\times\mathbb{R}^d)}:=\int_0^T\hat{\mathbb{E}}\left[\sup_{v\in\mathcal{V}}\int_{\mathbb{R}^d}K^2(u,z)v(dz)\right]du.$$

7.1. SDE's driven by G-Lévy processes

We will consider the following SDE driven by the d-dimensional G-Brownian motion B and the d-dimensional pure jump G-Lévy process L

$$dY_s^i=b^i(s,Y_s)ds+h^i(s,Y_s)\colon d\langle B\rangle_s+\sigma^i(s,Y_s)\cdot dB_s+\int_{\mathbb{R}^d}K^i(s,Y_{s-},z)L(dz,ds),\quad (13)$$

where $i=1,\dots,n$, $Y=(Y^1,\dots,Y^n)$. Denote $b=(b^1,\dots,b^n)$, $h=(h^1,\dots,h^n)$, $\sigma=(\sigma^1,\dots,\sigma^n)$ and $K=(K^1,\dots,K^n)$.

We will work under following standard assumptions.

Assumption 2. 1. *$b\colon[0,T]\times\mathbb{R}^n\times\Omega\to\mathbb{R}^n$ is Lipschitz continuous w.r.t. x uniformly w.r.t. (t,ω) (i.e. $|b(t,x)-b(t,y)|\leq c|x-y|$) and $b(.,x)\in\hat{\mathcal{M}}^2_G(0,T)$ for each $x\in\mathbb{R}^n$.*

2. $\sigma\colon [0,T]\times\mathbb{R}^n\times\Omega\to\mathbb{R}^{d\times n}$ *is Lipschitz continuous w.r.t.* x *uniformly w.r.t.* (t,ω) *(i.e.* $|\sigma(t,x)-\sigma(t,y)|\leq c|x-y|$*) and each row of* $\sigma(.,x)$ *belongs to* $\hat{\mathcal{M}}^2_G(0,T)$ *for each* $x\in\mathbb{R}^n$*.*
3. $K\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}^d\times\Omega\to\mathbb{R}^n$ *is Lipschitz continuous w.r.t.* x *uniformly w.r.t.* (t,ω,z) *(i.e.* $|K(t,x,z)-K(t,y,z)|\leq c|x-y|$*) and* $K(.,x,.)\in\hat{\mathcal{H}}^2_G([0,T]\times\mathbb{R}^d)$ *for each* $x\in\mathbb{R}^n$*.*
4. $h\colon [0,T]\times\mathbb{R}^n\times\Omega\to(\mathbb{S}^d)^{\times n}$ *is Lipschitz continuous w.r.t.* x *uniformly w.r.t.* (t,ω) *(i.e.* $|h(t,x)-h(t,y)|\leq c|x-y|$*) and* $h(.,x)$ *for each* $x\in\mathbb{R}^n$ *is a symmetric* $d\times d$ *matrix with each element taking values in* $\hat{\mathcal{M}}^2_G(0,T)$*.*

Definition 10. *The solution of the SDE (13) with the initial condition* $y_0\in\mathbb{R}^n$ *is the process* $Y\in\hat{\mathcal{M}}^2_G(0,T)$*, satisfying*

$$\begin{aligned}Y^i_t =&y_0+\int_0^t b^i(s,Y_s)ds+\int_0^t h^i(s,Y_s)\colon d\langle B\rangle_s\\&+\int_0^t\sigma^i(s,Y_s)\cdot dB_s+\int_0^t\int_{\mathbb{R}^d}K^i(s,Y_{s-},z)L(dz,ds).\end{aligned}$$

Theorem 28. *Under the Assumption 2 there exists the unique solution of the SDE (13) with the initial condition* $y_0\in\mathbb{R}^n$*.*

Proof. The proof is standard. We introduce the mapping

$$\Lambda\colon \hat{\mathcal{M}}^2_G(0,T)\to\hat{\mathcal{M}}^2_G(0,T)$$

by setting $\Lambda^i_t,\ t\in[0,T],\ i=1,\dots,n$ as

$$\begin{aligned}\Lambda^i_t(Y):=&y_0+\int_0^t b^i(s,Y_s)ds+\int_0^t h^i(s,Y_{s-})\colon d\langle B\rangle_s\\&+\int_0^t\sigma^i(s,Y_{s-})\cdot dB_s+\int_0^t\int_{\mathbb{R}^d}K^i(s,Y_{s-},z)L(dz,ds).\end{aligned}$$

By the Lipschitz continuity of the coefficients we easily obtain the following estimate

$$\begin{aligned}&\hat{\mathbb{E}}\left[|\Lambda_t(Y)-\Lambda_t(Y')|^2\right]\leq C\hat{\mathbb{E}}\left[\int_0^t|b(s,Y_s)-b(s,Y'_s)|^2ds\right]\\&+C\hat{\mathbb{E}}\left[\left|\int_0^t(h(s,Y_s)-h(s,Y'_s))\colon d\langle B\rangle_s\right|^2\right]+C\hat{\mathbb{E}}\left[\left|\int_0^t(\sigma(s,Y_s)-\sigma(s,Y'_s))\cdot dB_s\right|^2\right]\\&+C\hat{\mathbb{E}}\left[\left|\int_0^t\int_{\mathbb{R}^d}(K(s,Y_{s-},z)-K(s,Y'_{s-},z))L(dz,ds)\right|^2\right]\\&\leq C\int_0^t\hat{\mathbb{E}}\left[|b(s,Y_s)-b(s,Y'_s)|^2\right]ds+C\hat{\mathbb{E}}\left[\int_0^t|h(s,Y_s)-h(s,Y'_s)|^2ds\right]\\&+C\int_0^t\hat{\mathbb{E}}\left[|\sigma(s,Y_s)-\sigma(s,Y'_s)|^2\right]ds\\&+C\int_0^t\hat{\mathbb{E}}\left[\sup_{v\in\mathcal{V}}\int_{\mathbb{R}^d}|K(s,Y_{s-},z)-K(s,Y'_{s-},z)|^2v(dz)\right]ds\\&\leq C\int_0^t\hat{\mathbb{E}}\left[|Y_s-Y'_s|^2\right]ds,\quad t\in[0,T].\end{aligned}$$

We have applied here some standard inequalities, domination of the quadratic covariation differential by dt, the continuity of the stochastic integrals w.r.t. the appropriate norms and the Lipschitz continuity of the coefficients. Note that he constant C may vary from line to line, but depend only on the Lipschitz constant, dimensions d and n, time horizon T and the set $\mathcal{U}$.

Now multiplying the both sides od inequality above by e^{-2Ct} and integrating them on $[0,T]$, one gets

$$\begin{aligned}\int_0^T \hat{\mathbb{E}}\left[|\Lambda_t(Y)-\Lambda_t(Y')|^2\right]e^{-2Ct}dt &\leq C\int_0^T e^{-2Ct}\int_0^t \hat{\mathbb{E}}\left[|Y_s-Y'_s|^2\right]dsdt \\ &\leq C\int_0^T\int_s^T e^{-2Ct}\hat{\mathbb{E}}\left[|Y_s-Y'_s|^2\right]dtds \\ &= \frac{1}{2}\int_0^T (e^{-2Cs}-e^{-2CT})\hat{\mathbb{E}}\left[|Y_s-Y'_s|^2\right]ds.\end{aligned}$$

Thus we have the following inequality

$$\int_0^T \hat{\mathbb{E}}\left[|\Lambda_t(Y)-\Lambda_t(Y')|^2\right]e^{-2Ct}dt \leq \frac{1}{2}\int_0^T \hat{\mathbb{E}}\left[|Y_t-Y'_t|^2e^{-2Ct}\right]dt,$$

This equality shows that Λ is a contraction mapping on $\hat{\mathcal{M}}_G^2(0,T)$ equipped with the norm

$$\left(\int_0^T \hat{\mathbb{E}}\left[|Y_t|^2e^{-2Ct}\right]dt\right)^{1/2},$$

which is equivalent to the norm $\|.\|_{\hat{\mathcal{M}}_G^2(0,T)}$. As a consequence there exists a unique fixed point of Λ, which is the solution of our SDE. □

7.2. BSDE's and decoupled FBSDE's

We will consider the following type of BSDE:

$$dY_t^i = \hat{\mathbb{E}}\left[\xi^i + \int_t^T b^i(s,Y_s)ds + \int_t^T h^i(s,Y_s)\colon d\langle B\rangle_s \Big| \Omega_t\right], \quad t\in[0,T], \tag{14}$$

where $i=1,\dots,n$, $Y=(Y^1,\dots,Y^n)$ and $\xi=(\xi^1,\dots,\xi^n)$. Denote $b=(b^1,\dots,b^n)$ and $h=(h^1,\dots,h^n)$.

We will work under following standard assumptions.

Assumption 3. 1. $\xi^i\in L_G^1(\Omega_T)$.

2. *$b\colon[0,T]\times\mathbb{R}^n\times\Omega\to\mathbb{R}^n$ is Lipschitz continuous w.r.t. x uniformly w.r.t. (t,ω) (i.e. $|b(t,x)-b(t,y)|\leq c|x-y|$) and $b(.,x),\sigma(.,x)\in\hat{\mathcal{M}}_G^1(0,T)$ for each $x\in\mathbb{R}^n$.*
3. *$h\colon[0,T]\times\mathbb{R}^n\times\Omega\to(\mathbb{S}^d)^{\times n}$ is Lipschitz continuous w.r.t. x uniformly w.r.t. (t,ω) (i.e. $|h(t,x)-h(t,y)|\leq c|x-y|$) and $h(.,x)$ for each $x\in\mathbb{R}^n$ is a random matrix such that each element belongs to $\hat{\mathcal{M}}_G^1(0,T)$.*

Definition 11. *The solution of the BSDE (14) is the process $Y\in\hat{\mathcal{M}}_G^1(0,T)$, satisfying eq. (14).*

Theorem 29. *Under the Assumption 3 there exists a unique solution of the BSDE (14).*

Proof. The proof is very similar to the SDE case and is nearly identical to the proof for the G-Brownian motion case (see Theorem V.2.2 in [10]), so we omit it. $\square$

As the concequence, we can introduce decoupled FBSDE's. For simplicity assume $n = 1$.

Consider the following SDE

$$\begin{aligned} dX_s^{t,\xi} &= b(X_s^{t,\xi})ds + h(X_s^{t,\xi})\colon d\langle B\rangle_s + \sigma(X_s^{t,\xi})\cdot dB_s + \int_{\mathbb{R}^d} K(X_{s-}^{t,\xi}, z)L(dz, ds), \\ X_t^{t,\xi} &= \xi, \quad s \in [t, T], \end{aligned} \tag{15}$$

where $\xi \in L_G^2(\Omega_t)$, $\sigma\colon \mathbb{R} \to \mathbb{R}^d$, $b\colon \mathbb{R} \to \mathbb{R}$, $h\colon \mathbb{R} \to \mathbb{S}^d$ and $K\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ are deterministic and Lipschitz continuous functions w.r.t. (x, y) (and uniformly in z for K). By Theorem 28, there is a unique solution of this SDE. Now consider associated BSDE

$$Y_s^{t,\xi} = \hat{\mathbb{E}}\left[\Phi(X_T^{t,\xi}) + \int_s^T f(X_r^{t,\xi}, Y_r^{t,\xi})dr + \int_s^T g(X_r^{t,\xi}, Y_r^{t,\xi})d\langle B\rangle_r \Big| \Omega_s\right], \quad s \in [t, T], \tag{16}$$

where $\Phi\colon \mathbb{R} \to \mathbb{R}$, $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R}\times\mathbb{R} \to \mathbb{S}^d$ are deterministic and Lipschitz continuous functions w.r.t. (x, y). Theorem 29 guarantees the existence and uniqueness of the solution. The pair $(X_s^{t,\xi}, Y_s^{t,\xi})$ is the decoupled FBSDE with jumps.

In the forthcoming work we will deal with the optimal control of these systems driven by the G-Lévy processes with finite activity.

Appendix

Proof of Theorem 6. Let $\psi \in C_b^{2,3}([0, T]\times\mathbb{R}^d)$ be such that $\psi \geq u$ and for a fixed $(t, x) \in [0, T]\times\mathbb{R}^d$ we have $\psi(t, x) = u(t, x)$.

By the Itô formula we have

$$\begin{aligned} \psi\left(t+h, x + B_{t+h}^{t,\theta}\right) - \psi(t, x) &= \int_t^{t+h} \frac{\partial\psi}{\partial s}\left(s, x + B_{s-}^{t,\theta}\right) ds \\ &+ \int_t^{t+h} \langle D\psi\left(s, x + B_{s-}^{t,\theta}\right), \theta_s^{1,c}\rangle ds + \int_t^{t+h} \langle D\psi\left(s, x + B_{s-}^{t,\theta}\right), \theta_s^{2,c} dW_s\rangle \\ &+ \int_t^{t+h} \frac{1}{2}\operatorname{tr}\left[\theta_s^{2,c}\left(\theta_s^{2,c}\right)^T D^2\psi\left(s, x + B_{s-}^{t,\theta}\right)\right] ds \\ &+ \int_t^{t+h}\int_{\mathbb{R}} \left[\psi\left(s, x + B_{s-}^{t,\theta} + \theta_s^d z\right) - \psi\left(s, x + B_{s-}^{t,\theta}\right)\right] N(ds, dz), \end{aligned}$$

where $N(ds, dz)$ is a Poisson random measure associated with N. Taking $\mathbb{E}^{\mathbb{P}}$ we get the following

$$\mathbb{E}^{\mathbb{P}}\left[\psi\left(t+h, x + B_{t+h}^{t,\theta}\right) - \psi(t, x)\right] =: \mathbb{E}^{\mathbb{P}}[I_1] + \mathbb{E}^{\mathbb{P}}[I_2] + \mathbb{E}^{\mathbb{P}}[I_3] + \mathbb{E}^{\mathbb{P}}[I_4] + \mathbb{E}^{\mathbb{P}}[I_5].$$

Note that $\mathbb{E}^{\mathbb{P}}[I_3] = 0$. Moreover, $\left\{\frac{\partial\psi}{\partial s} + \langle D\psi, \theta_s^{1,c}\rangle + \frac{1}{2}\operatorname{tr}\left[\theta_s^{2,c}\left(\theta_s^{2,c}\right)^T D^2\psi\right]\right\}(s,y)$ is uniformly Lipschitz continuous in (s,y). Note also the following estimate

$$\mathbb{E}^{\mathbb{P}}[|B_{t+h}^{t,\theta}|] \le C[h^{1/2}+h]h.$$

Combining those two results we get the following estimate (the constant C may vary from line to line)

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[I_1+I_2+I_4\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\int_t^{t+h}\left\{\frac{\partial\psi}{\partial s} + \langle D\psi, \theta_s^{1,c}\rangle + \frac{1}{2}\operatorname{tr}\left[\theta_s^{2,c}\left(\theta_s^{2,c}\right)^T D^2\psi\right]\right\}\left(s, x+B_{s-}^{t,\theta}\right)ds\right] \\
&\le \mathbb{E}^{\mathbb{P}}\left[\int_t^{t+h}\left\{\frac{\partial\psi}{\partial s} + \langle D\psi, \theta_s^{1,c}\rangle + \frac{1}{2}\operatorname{tr}\left[\theta_s^{2,c}\left(\theta_s^{2,c}\right)^T D^2\psi\right]\right\}(t,x)\,ds\right] \\
&+ C(h^{1/2}+h).
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[I_5\right] &= \mathbb{E}^{\mathbb{P}}\left[\int_t^{t+h}\int_{\mathbb{R}}\left[\psi\left(s, x+B_{s-}^{t,\theta}+\theta_s^d z\right) - \psi\left(s, x+B_{s-}^{t,\theta}\right)\right]N(ds,dz)\right] \\
&= \int_t^{t+h}\mathbb{E}^{\mathbb{P}}\left[\psi\left(t, x+B_{s-}^{t,\theta}+\theta_s^d\right) - \psi\left(s, x+B_{s-}^{t,\theta}\right)\right]ds \\
&\le \int_t^{t+h}\mathbb{E}^{\mathbb{P}}\left[\psi\left(t, x+\theta_s^d\right) - \psi(t,x)\right]ds + C(h+h^{1/2})h \\
&=\le \int_t^{t+h}\int_{\mathbb{R}^d}\left[\psi\left(t,x+z\right)-\psi(t,x)\right]\mu_{\theta_s^d}(dz)ds + C(h+h^{1/2})h.
\end{aligned}$$

Connecting those two estimates with Theorem 5 we get

$$\begin{aligned}
0 &= \sup_{\theta\in\mathcal{A}_{t,t+h}^{\mathcal{U}}}\mathbb{E}^{\mathbb{P}}\left[u(t+h, x+B_{t+h}^{t,\theta}) - u(t,x)\right] \\
&\le \sup_{\theta\in\mathcal{A}_{t,t+h}^{\mathcal{U}}}\mathbb{E}^{\mathbb{P}}\left[\psi(t+h, x+B_{t+h}^{t,\theta}) - \psi(t,x)\right] \\
&\le \sup_{\theta\in\mathcal{A}_{t,t+h}^{\mathcal{U}}}\mathbb{E}^{\mathbb{P}}\left[\int_t^{t+h}\left\{\frac{\partial\psi}{\partial s} + \langle D\psi, \theta_s^{1,c}\rangle + \frac{1}{2}\operatorname{tr}\left[\theta_s^{2,c}\left(\theta_s^{2,c}\right)^T D^2\psi\right]\right\}(t,x)\,ds\right. \\
&+ \left.\int_t^{t+h}\int_{\mathbb{R}^d}\left[\psi\left(t,x+z\right)-\psi(t,x)\right]\mu_{\theta_s^d}(dz)ds\right] + C(h+h^{1/2})h \\
&\le \mathbb{E}^{\mathbb{P}}\left[\int_t^{t+h}\sup_{(p,Q)\in\mathcal{P}\times\mathcal{Q}}\left\{\frac{\partial\psi}{\partial s} + \langle D\psi, p\rangle + \frac{1}{2}\operatorname{tr}\left[QQ^T D^2\psi\right]\right\}ds\right. \\
&+ \left.\int_t^{t+h}\sup_{v\in\mathcal{V}}\int_{\mathbb{R}^d}\left[\psi\left(t,x+z\right)-\psi(t,x)\right]v(dz)ds\right] + C(h+h^{1/2})h
\end{aligned}$$

$$= \left\{ \sup_{(v,p,Q)\in\mathcal{U}} \left[\frac{\partial\psi}{\partial s} + \langle D\psi, p\rangle + \frac{1}{2}\operatorname{tr}\left[QQ^T D^2\psi\right] \right.\right.$$

$$\left.\left. + \int_{\mathbb{R}^d} \left[\psi\left(t, x+z\right) - \psi(t,x)\right] v(dz) \right] + C(h + h^{1/2}) \right\} \cdot h$$

Thus dividing both sides by h and going with it to 0, we see that u is the viscosity subsolution of the integropartial PDE in Theorem 6. By the same argument one can prove that u is also a supersolution. □

Proof of Theorem 7. We will use the inductive argument. First, we prove it for $n = 1$, i.e.

$$\xi = \phi(X_{t_1} - X_{t_0}), \quad \phi \in C_{b,Lip}(\mathbb{R}^d).$$

We need to prove that

$$\hat{\mathbb{E}}[\phi(X_{t_1} - X_{t_0})] = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{t_0,\theta}_{t_1})\right] = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_0,t_1}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{t_0,\theta}_{t_1})\right].$$

By the construction of the sublinear expectation $\hat{\mathbb{E}}[.]$ we know that LHS is equal to

$$\hat{\mathbb{E}}[\phi(X_{t_1} - X_{t_0})] = v(t_1 - t_0, 0),$$

where v is a unique viscosity solution of the integro-PDE 2 with initial condition $v(0,x) = \phi(x)$. However note that the function $(t,x) \mapsto u(T-t,x)$ also solves the same integro-PDE. Thus $u(T-t,x) = v(t,x)$ by the uniqueness of the solution. Thus we need to show that

$$u(T - (t_1 - t_0), 0) = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{t_0,\theta}_{t_1})\right] = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_0,t_1}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{t_0,\theta}_{t_1})\right].$$

By the definition of u and Yan's commutation theorem one gets

$$\begin{aligned}
u(T - (t_1 - t_0), 0) &= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{T-(t_1-t_0),T}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{T-(t_1-t_0),\theta}_T)\right] \\
&= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{T-(t_1-t_0),T}} \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\phi(B^{T-(t_1-t_0),\theta}_T)\big|\mathcal{F}_{T-(t_1-t_0)}\right]\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{T-(t_1-t_0),T}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{T-(t_1-t_0),\theta}_T)\big|\mathcal{F}_{T-(t_1-t_0)}\right]\right]
\end{aligned}$$

Use Lemma 9, property 3 with $x = 0$ to get

$$\begin{aligned}
u(T - (t_1 - t_0), 0) &= \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{T-(t_1-t_0),T}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{T-(t_1-t_0),\theta}_T)\big|\mathcal{F}_{T-(t_1-t_0)}\right]\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_0,t_1}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{t_0,\theta}_{t_1})\big|\mathcal{F}_{t_0}\right]\right] = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_0,t_1}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{t_0,\theta}_{t_1})\right].
\end{aligned}$$

Note however that we can also use Lemma 12, property 3 with $\zeta = 0$, before using the commutation theorem and then we get

$$u(T-(t_1-t_0),0) = \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_0,t_1}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{t_0,\theta}_{t_1})\big|\mathcal{F}_{t_0}\right]\right] = \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,t_0,t_1}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{t_0,\theta}_{t_1})\big|\mathcal{F}_{t_0}\right]\right]$$
$$= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,t_0,t_1}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{t_0,\theta}_{t_1})\right] = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,T}} \mathbb{E}^{\mathbb{P}}\left[\phi(B^{t_0,\theta}_{t_1})\right].$$

Thus we have proved the theorem for $n = 1$.

Now assume that we have the desired representation for $n-1$. We have to prove that

$$\hat{\mathbb{E}}\left[\phi\left(X_{t_1}-X_{t_0},\dots,X_{t_n}-X_{t_{n-1}}\right)\right] = \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,\infty}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(B^{t_0,\theta}_{t_1},\dots,B^{t_{n-1},\theta}_{t_n}\right)\right]$$
$$= \sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{\pi}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(B^{t_0,\theta}_{t_1},\dots,B^{t_{n-1},\theta}_{t_n}\right)\right],$$

where $\pi = \{t_0,t_1,\dots,t_n\}$.

By the construction of $\hat{\mathbb{E}}[.]$ we know that LHS is equal to

$$\hat{\mathbb{E}}[\phi(X_{t_1}-X_{t_0},\dots,X_{t_n}-X_{t_{n-1}})] = \hat{\mathbb{E}}[\psi_1(X_{t_1}-X_{t_0},\dots,X_{t_{n-1}}-X_{t_{n-2}})],$$

where $\psi_1(x_1,\dots,x_{n-1}) = \hat{\mathbb{E}}[\phi(x_1,\dots,x_{n-1},,X_{t_{n-1}}-X_{t_{n-2}})]$. Hence by substitution and the inductive assumption (used twice) we get

$$\begin{aligned}
&\hat{\mathbb{E}}[\psi_1(X_{t_1}-X_{t_0},\dots,X_{t_{n-1}}-X_{t_{n-2}})]\\
&\quad= \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[\phi(x,X_n-X_{n-1})]|_{x=(X_{t_1}-X_{t_0},\dots,X_{t_{n-1}}-X_{t_{n-2}})}\right]\\
&\quad= \sup_{\theta'\in\mathcal{A}^{\mathcal{U}}_{\pi'}} \mathbb{E}^{\mathbb{P}}\left[\left\{\sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_{n-1},t_n}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(x,B^{t_{n-1},\theta}_{t_n}\right)\right]\right\}\Big|_{x=\left(B^{t_0,\theta'}_{t_1},\dots,B^{t_{n-2},\theta'}_{t_{n-1}}\right)}\right]\\
&\quad= \sup_{\theta'\in\mathcal{A}^{\mathcal{U}}_{\pi'}} \mathbb{E}^{\mathbb{P}}\left[\left\{\sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_{n-1},t_n}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(x,B^{t_{n-1},\theta}_{t_n}\right)|\mathcal{F}_{t_{n-1}}\right]\right\}\Big|_{x=\left(B^{t_0,\theta'}_{t_1},\dots,B^{t_{n-2},\theta'}_{t_{n-1}}\right)}\right],
\end{aligned}$$

where $\pi' = \pi\setminus\{t_n\}$. Now using Lemma 11 we can write the latter as

$$\sup_{\theta'\in\mathcal{A}^{\mathcal{U}}_{\pi'}} \mathbb{E}^{\mathbb{P}}\left[\left\{\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_{n-1},t_n}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(x,B^{t_{n-1},\theta}_{t_n}\right)|\mathcal{F}_{t_{n-1}}\right]\right\}\Big|_{x=\left(B^{t_0,\theta'}_{t_1},\dots,B^{t_{n-2},\theta'}_{t_{n-1}}\right)}\right]$$
$$= \sup_{\theta'\in\mathcal{A}^{\mathcal{U}}_{\pi'}} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_{n-1},t_n}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(B^{t_0,\theta'}_{t_1},\dots,B^{t_{n-2},\theta'}_{t_{n-1}},B^{t_{n-1},\theta}_{t_n}\right)|\mathcal{F}_{t_{n-1}}\right]\right],$$

so using Yan's commutation theorem we get the second equality in the theorem.

We can get the first inequality similarly. By inductive argument and Lemma 11 we get that

$$\begin{aligned}
&\hat{\mathbb{E}}[\psi_1(X_{t_1}-X_{t_0},\dots,X_{t_{n-1}}-X_{t_{n-2}})] \\
&\quad= \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[\phi(x,X_n-X_{n-1})]|_{x=(X_{t_1}-X_{t_0},\dots,X_{t_{n-1}}-X_{t_{n-2}})}\right] \\
&\quad= \sup_{\theta'\in\mathcal{A}^{\mathcal{U}}_{0,t_{n-1}}} \mathbb{E}^{\mathbb{P}}\left[\left\{\sup_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_{n-1},t_n}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(x,B^{t_{n-1},\theta}_{t_n}\right)\right]\right\}\Big|_{x=\left(B^{t_0,\theta'}_{t_1},\dots,B^{t_{n-2},\theta'}_{t_{n-1}}\right)}\right] \\
&\quad= \sup_{\theta'\in\mathcal{A}^{\mathcal{U}}_{0,t_{n-1}}} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_{n-1},t_n}} \mathbb{E}^{\mathbb{P}}\left[\phi\left(B^{t_0,\theta'}_{t_1},\dots,B^{t_{n-2},\theta'}_{t_{n-1}},B^{t_{n-1},\theta}_{t_n}\right)|\mathcal{F}_{t_{n-1}}\right]\right].
\end{aligned}$$

Using now Lemma 12, property 3, we get that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{t_{n-1},t_n}} \mathbb{E}^{\mathbb{P}}[\phi(B^{t_0,\theta'}_{t_1},\dots,B^{t_{n-2},\theta'}_{t_{n-1}},B^{t_{n-1},\theta}_{t_n})|\mathcal{F}_{t_{n-1}}] \\
&= \operatorname*{ess\,sup}_{\theta\in\mathcal{A}^{\mathcal{U}}_{0,t_{n-1},t_n}} \mathbb{E}^{\mathbb{P}}[\phi(B^{t_0,\theta'}_{t_1},\dots,B^{t_{n-2},\theta'}_{t_{n-1}},B^{t_{n-1},\theta}_{t_n})|\mathcal{F}_{t_{n-1}}],
\end{aligned}$$

and we get the first equality by combining this result with the commutation theorem. □

Proof of Lemma 10. First, we will prove that for any $B\in\mathcal{F}$ and any family $\{X_i\}_{i\in I}$ of bounded random variables one has

$$\left(\operatorname*{ess\,sup}_{i\in I}{}^{\mathbb{P}} X_i\right)\Big|_B = \operatorname{ess\,sup}^{\mathbb{P}|_B}(X_i|_B), \tag{17}$$

where $\mathbb{P}|_B$ is a conditional probability on B. This property follows directly from the definition of essential supremum. Namely, since $\operatorname{ess\,sup}^{\mathbb{P}}\{X_i\colon i\in I\}\geq X_j$ $\mathbb{P}$-a.s. for all $j\in I$ one also has that

$$\left(\operatorname*{ess\,sup}_{i\in I}{}^{\mathbb{P}} X_i\right)\Big|_B \geq X_j|_B \quad \mathbb{P}|_B-a.s., \quad j\in I.$$

Take now any Z such that $Z\geq X_j|_B$ $\mathbb{P}|_B$-a.s. for all $j\in I$. Define $Z':=Z\,\mathbb{1}_B+\infty\,\mathbb{1}_{B^c}$. Then of course $Z'\geq X_j$ $\mathbb{P}$-a.s. for all $j\in I$. Then

$$Z'\geq \operatorname*{ess\,sup}_{i\in I}{}^{\mathbb{P}} X_i,\ \mathbb{P}-a.s. \implies Z=Z'|_B\geq \left(\operatorname*{ess\,sup}_{i\in I}{}^{\mathbb{P}} X_i\right)\Big|_B,\ \mathbb{P}|_B-a.s.$$

Thus by the definition of the essential supremum we have eq. 17.

As the next step note that the assertion above gives us the following property: if we have two families $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ of bounded random variables such that $X_i=Y_i$ $\mathbb{P}$-a.s. on $B\in\mathcal{F}$ for all $i\in I$ then

$$\operatorname*{ess\,sup}_{i\in I} X_i = \operatorname*{ess\,sup}_{i\in I} Y_i,\ \mathbb{P}-a.s. \text{ on } B.$$

But this property gives nearly directly the assertion of the lemma. □

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